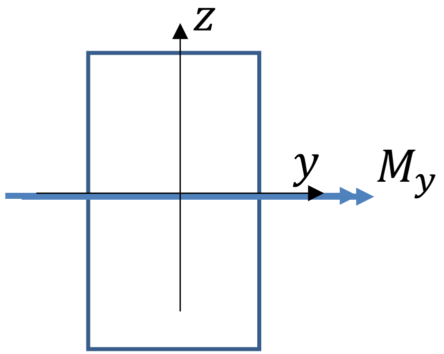
Strength of Materials

3. Biaxial bending

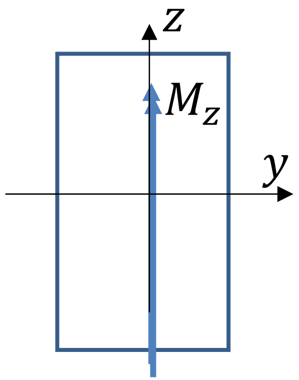
Biaxial bending - definition

(y, z) – principal central inertia axes



$$\sigma_x = \frac{M_y}{I_y} z$$

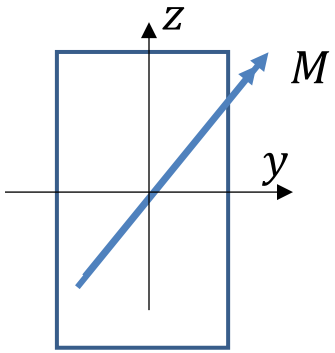
simple bending
about y axis



$$\sigma_x = -\frac{M_z}{I_z} y$$

tension for y < 0

simple bending
about z axis



$$\sigma_x = \frac{M_y}{I_y} z - \frac{M_z}{I_z} y$$

biaxial bending
(inclined bending)
(unsymmetrical bending)

stress distribution is given by the equation of plane
neutral axis: intersection of the c-s plane and the stress plane
neutral axis is a straight line that passes by the c-s centroid

are the directions of bending moment and neutral axis the same?

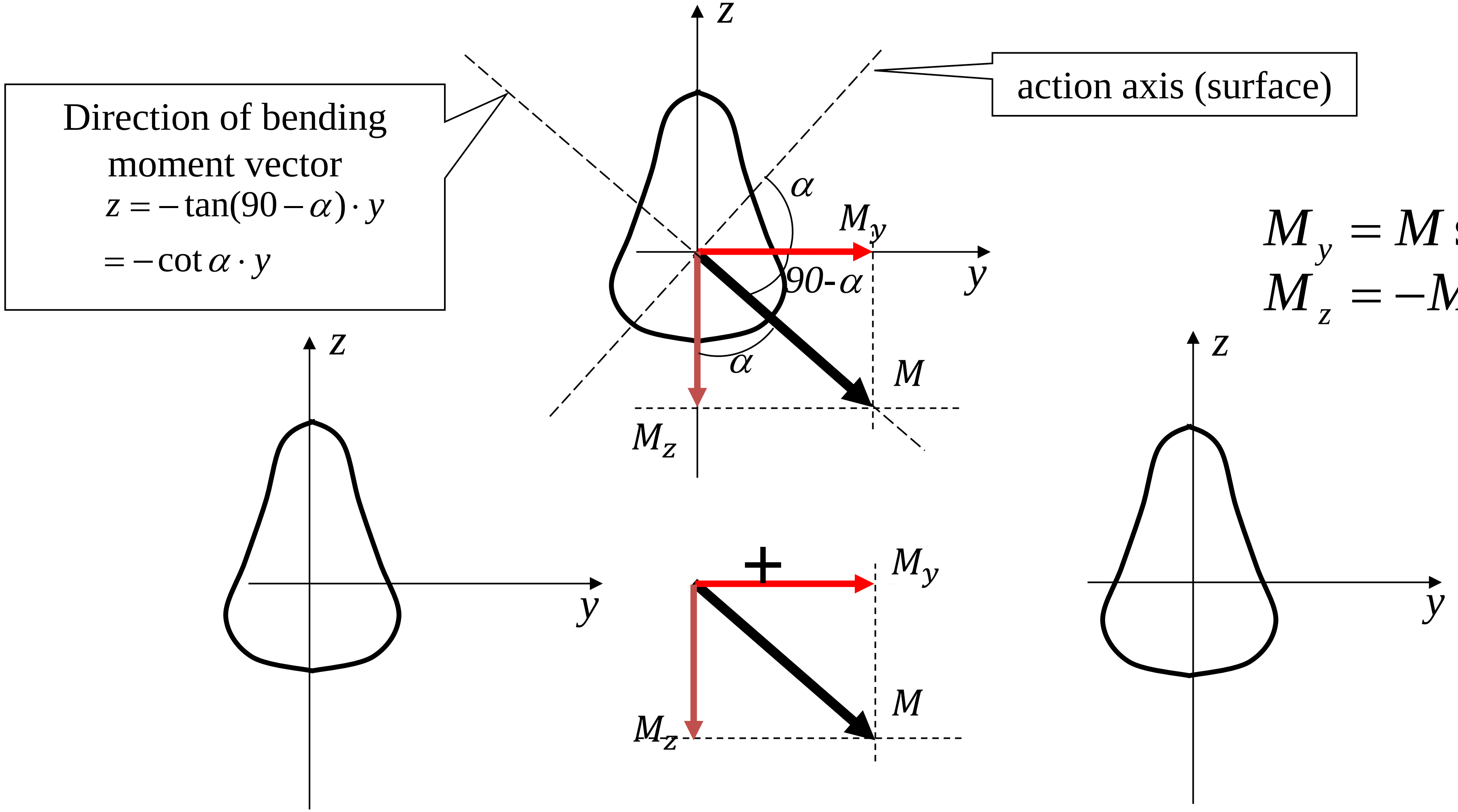
$$\sigma_x = 0 \rightarrow z = \frac{M_z}{M_y} \frac{I_y}{I_z} y =$$

$$= \tan\alpha \cdot \frac{I_y}{I_z} y$$

never! (only if $I_y = I_z$, but then every central axis is principal too, so it is a simple bending!)

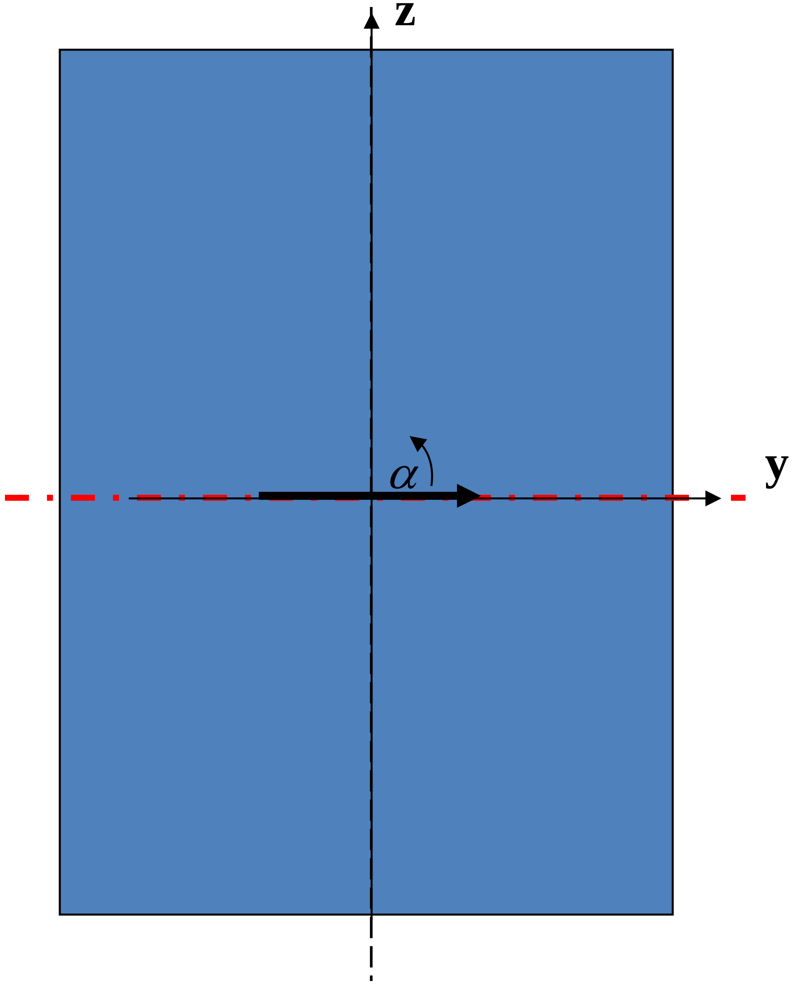
Biaxial bending – definition cont.

If bending moment does not coincide with principal central axis of c.-s. inertia we have to deal with **biaxial bending**.



$$M_y = M\sin\alpha$$
$$M_z = -M\cos\alpha$$

Biaxial bending – animation



h/b=1,41

J_y/J_z =2

Angle α

80º

70º

60º

50º

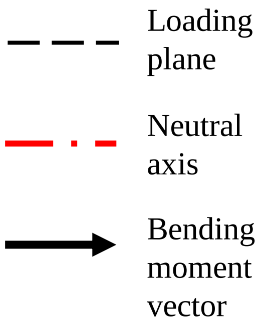
40º

30º

20º

10º

00º

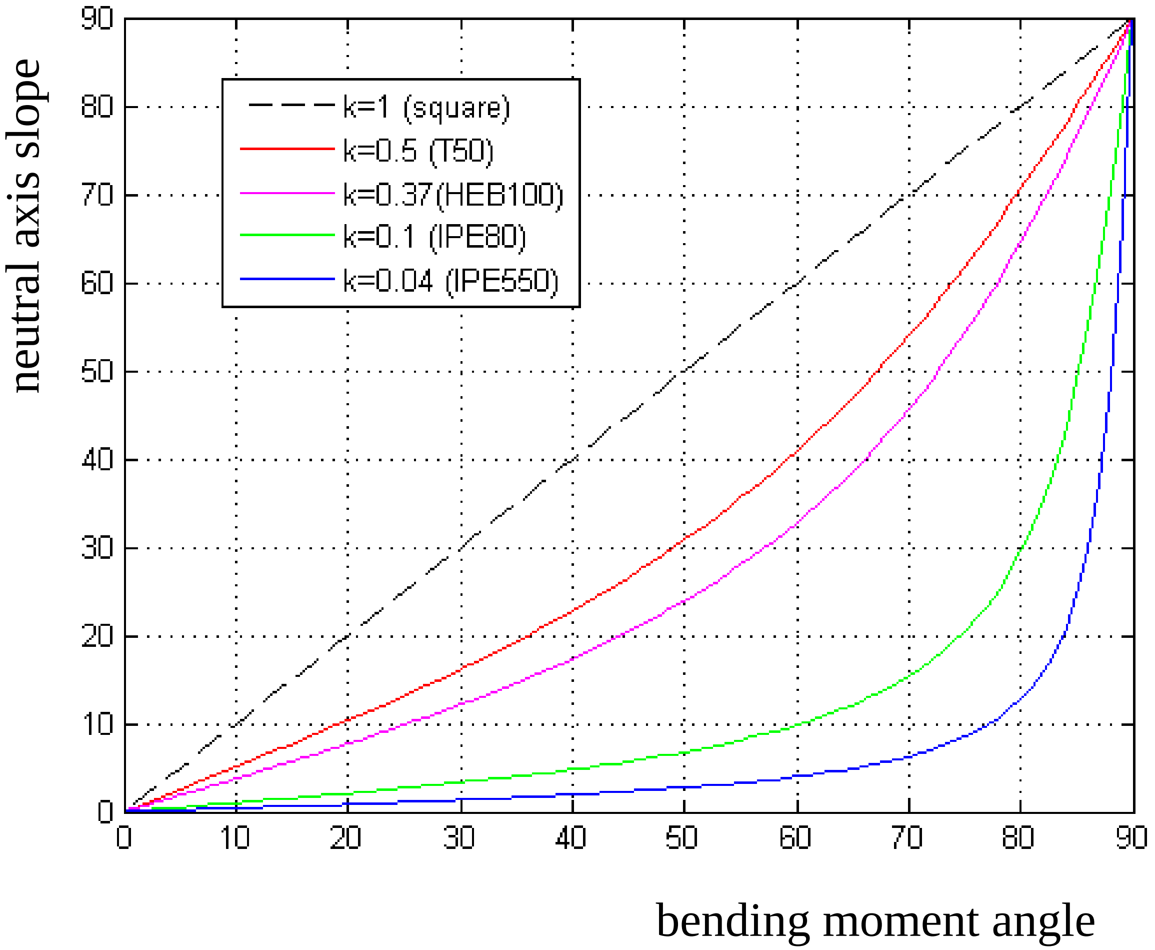


neutral axis deviates towards the axis of greatest flexibility

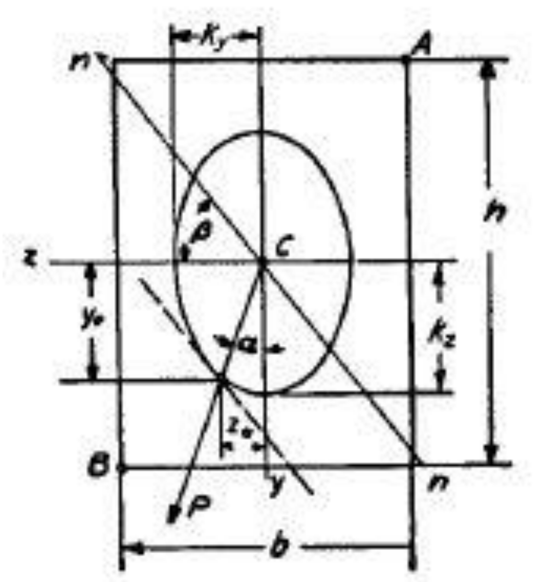
The neutral axis direction is located between the directions of bending moment and axis of minimal inertia moment

Neutral axis slope vs. bending moment angle

$$k \stackrel{\text{def}}{=} \frac{I_z}{I_y} \leq 1$$



Biaxial bending – ellipse of inertia



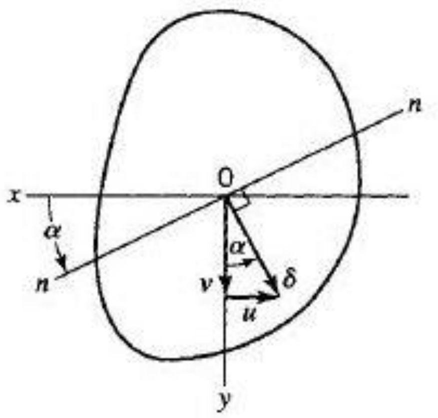
the neutral axis: $z = \tan\alpha \cdot \frac{I_y}{I_z} y,$ the inertia ellipse equation: $\frac{y^2}{i_z^2} + \frac{z^2}{i_y^2} = 1$

the equation of the tangent at the point (y_0, z_0): $\frac{yy_0}{i_z^2} + \frac{zz_0}{i_y^2} = 1$

the tangent of the angle between the *z* axis and this tangent will be:

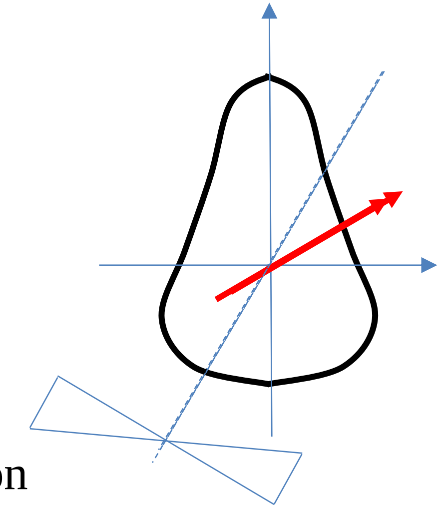
$$\frac{z_0}{y_0} \cdot \frac{i_z^2}{i_y^2} = \tan\alpha \frac{I_z}{I_y} = \tan\beta$$

The neutral axis is parallel to the tangent drawn to the ellipse of inertia at the point of intersection of this ellipse with the direction of loads axis.



The total deflection δ of the centroid at any section of the beam is perpendicular to the neutral axis.
The deflection produced by each of two components of the bending moment can easily be obtained by superposition (of the simple bendings).

Biaxial bending – limit strength state



max σ_x at the fibers the most distant from the neutral axis

$\max|\sigma_x| \leq R$
or
$\max \sigma_x \leq R_t$ for tension
$\min \sigma_x \geq R_c$ for compression

In many usual cross-sections, such as I-beams, C-channels, rectangular sections, etc., it is possible to identify the points of maximum values of the stress without having to compute the orientation of the neutral axis. It is evident, that the furthest point from the neutral axis is one of the corners. In these cases, we can use directly two section moduli:

$$|\sigma_{\max}| = \frac{|M_y|}{W_y} + \frac{|M_z|}{W_z}$$

Attention! The above formula is valid only in these particular cases. When the cross-section outline is not rectangular, the general, not simplified method, should be adopted and the neutral axis direction has to be found.

Biaxial bending – limit stress state cont.

The calculations are relatively simple for the cross-section with an axis of symmetry:

- reference axes
- position of the centroid (only one coordinate has to be found because of symmetry)
- usually, principal central axes are parallel to the reference axes
- relatively simple calculation of points coordinates in principal central coordinate system
- evaluation of the furthest points distance from the neutral axis (see remarks below)
- application of the stress formula at the chosen point(s)

$$\sigma_x(P) = \frac{M_y}{I_y} z_P - \frac{M_z}{I_z} y_P, \qquad \max|\sigma_x| \leq R, \qquad \text{or} \quad \sigma_x(P) \leq R_t, \qquad \sigma_x(P) \geq R_c$$

Tip: It often is easier to verify directly the stress at some points than to calculate and compare their distance from the neutral axis and verify the stress level at the furthest point.

Tip: It is a good idea to get a big picture of the stress distribution.

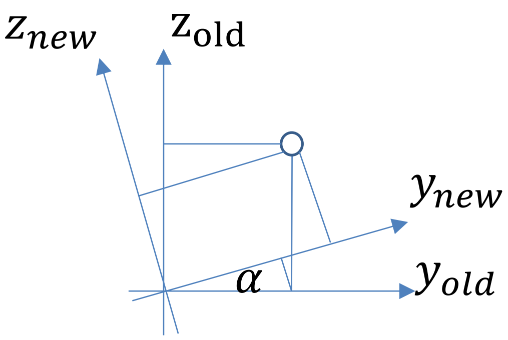
If the neutral axis passes through two points $1(x_1,y_1)$ and $2(x_2,y_2)$ then the distance of $P(x_P,y_P)$ from the axis is

$$d = \frac{|(y_2-y_1)x_p-(x_2-x_1)y_p+x_2y_1-y_2x_1|}{\sqrt{(y_2-y_1)^2+(x_2-x_1)^2}}$$

Biaxial bending – limit stress state cont.

If a cross-section has no symmetry axis, the calculations are much more complicated. Additively other steps are necessary:

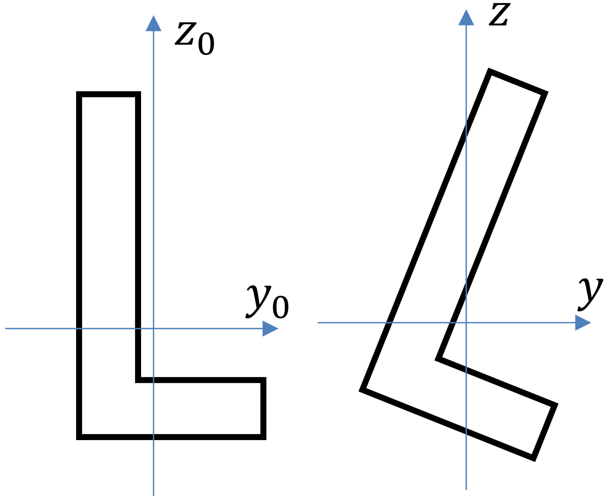
- calculation of both coordinates of the cross-section centroid
- calculation of central as well as principal directions and values
- calculation of bending moment components (resolution into the components, in the principal central c.s.)
- calculation of points coordinates (in the principal central coordinate set)



Transformation rule:

$$y_{new} = y_{old}\cos\alpha + z_{old}\sin\alpha$$
$$z_{new} = -y_{old}\sin\alpha + z_{old}\cos\alpha$$

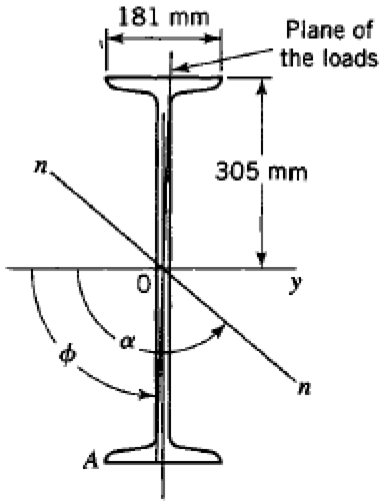
get a big picture and
sniff at the problem



before and after transformation

Biaxial bending – example

[Boresi] An I-beam 610×181 ($I_y = 93.7 \cdot 10^6$ mm^4 and $I_z = 0.187 \cdot 10^6$ mm^4) is subjected to a bending moment *M* in a plane with angle α = 1.5533 rad; (the plane of the loads is 1° (π/180 rad) clockwise from the (*y*, *z*) plane of symmetry.) Determine the neutral axis orientation and the ratio of the maximum tensile stress in the beam to the maximum tensile stress for symmetrical (i.e. simple) bending.



the bending moments $M_y = M\sin\varphi = 0.9998M, M_z = M\cos\varphi = -0.01745M$

neutral axis $\sigma_x = \frac{M_y}{I_y}z - \frac{M_z}{I_z}y = 0 \rightarrow z = \frac{M_z}{M_y}\frac{I_y}{I_z}y = -\cot 89° \frac{937}{18.7} = -0.8749$

$180 - \alpha = 41.18°$, way bigger than that of loads

max σ_x in the biaxial bending case

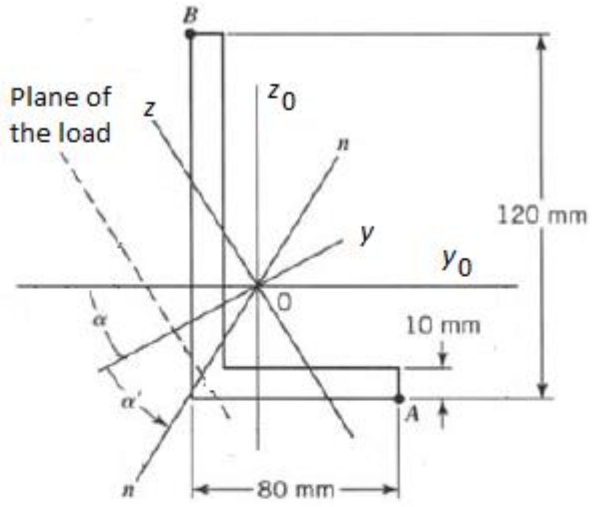
$$\sigma_x^{(1)} = \frac{0.9998M}{937 \cdot 10^6}305 + \frac{0.01745M}{18.7 \cdot 10^6}90.5 = 4.099 \cdot 10^{-7}M$$

max σ_x in the case of simple bending $\sigma_x^{(2)} = \frac{M}{I_y}z_{\max} = \frac{305}{937 \cdot 10^6}M = 3.255 \cdot 10^{-7}M$

$$\frac{\sigma_x^{(1)}}{\sigma_x^{(2)}} = 1.259$$

The maximum stress in the I-beam is increased 25.9% when the plane of the loads is merely 1° from the symmetrical vertical plane. This section should not be used as beam unless the lateral deflection is prevented.

Biaxial bending - example



Plates are welded together to form the 120 mm by 80 mm by 10 mm angle-section beam shown in the Figure. The section is subjected to a bending moment 4.8 kNm, that lies in the plane making an angle $\varphi = \frac{2}{3}\pi$ with the x-z plane. Determine the maximum tensile and compressive bending stresses.

Solution
cross-section characteristics $A = 19\ \mathrm{cm}^2, 0(1.974{,}3.974)$
$I_{y0} = 278.3\ \mathrm{cm}^4\ , I_{z0} = 100.3\ \mathrm{cm}^4, I_{y0z0} = -97.3\ \mathrm{cm}^4$
$I_y = 321.2\ \mathrm{cm}^4, I_z = 57.4\ \mathrm{cm}^4, \alpha = 23.78°$

the load axis angle is 120°, so the bending moment direction is 30° (from y_0 axis)
moments about principal central axes are
$M_y = -4.8 \cdot 10^3 \cdot \cos(30 - 23.78) = -4.772\ \mathrm{kNm}$, $M_z = -4.8 \cdot 10^3 \cdot \sin(30 - 23.78) = -0.520\ \mathrm{kNm}$
neutral axis $\sigma_x = 0 \rightarrow z = z = \frac{I_y}{I_z}\frac{M_z}{M_y} y = 0.6098y \rightarrow\ \alpha' = 31.37°$
points coordinates are $A(3.912, -6.066), B(\ 1.430, 8.141)$
normal stress at A: $\sigma_x^A = \frac{(-4772)\cdot(-0.06066)}{321.4\cdot 10^{-8}} - \frac{(-520)\cdot 0.03912}{57.4\cdot 10^{-8}} = 125.6$ MPa
normal stress at B: $\sigma_x^B = \frac{(-4772)\cdot 0.08141}{321.4\cdot 10^{-8}} - \frac{(-520)\cdot 0.0143}{57.4\cdot 10^{-8}} = -108.0$ MPa

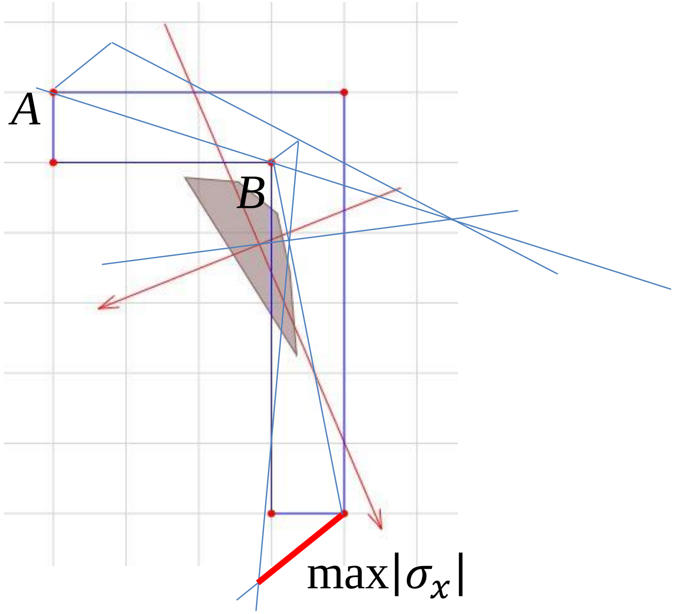
Biaxial bending - example

An angle cross-section 6×4×1 cm is subjected to biaxial bending. Given the stress values at two points: $\sigma_x^A = 40$ MPa, $\sigma_x^B = 20$ MPa, determine the value of $\max|\sigma_x|$.

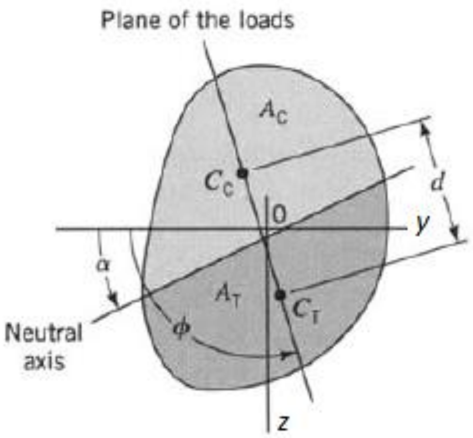
Solution

These three points define a stress surface.

We can use the stress equation or simply the proportions.



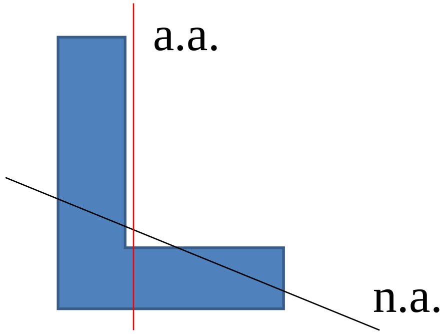
Biaxial bending – fully plastic load



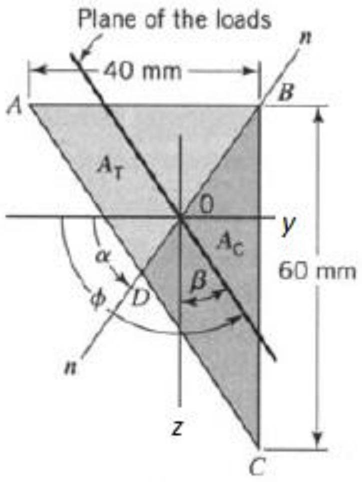
Prandtl's schematization with the yield point $\sigma_0$in both tension and compression
because $N = 0$, the neutral axis divides the cross-section in half, $A_t = A_c$
resultant of tension stresses of A_t is applied at the centroid C_t
resultant of compression stresses of $A_c$is applied at the centroid C_c
these resultants form a couple of forces: $M = \sigma_0 A_t d = \frac{\sigma_0 A d}{2}$
the centroids of the cross-section halves lie on the action axis

In contrast to the direct calculation of fully plastic load in simple bending case of the cross-section with a symmetry axis, an inverse method is required to determine the fully plastic load for a cross-section subjected to biaxial bending.
Although the plane of the loads is generally specified for a given cross-section, the orientation and location of the neutral axis must be determined by trial and error method.
We seek such position of the neutral axis that:

- it bisects the cross-section
- the centroids direction coincides with the load direction



Biaxial bending - example

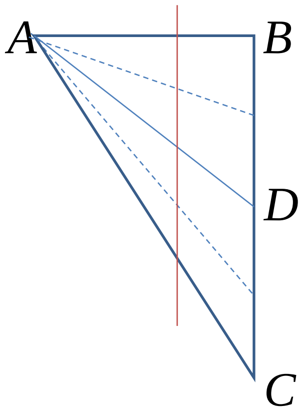


(Boresi, the a) part) A steel beam has the cross section shown in the Figure and is made of a steel having a yield point stress $\sigma_0 = 280$ MPa.
a) Determine the fully plastic moment for the condition that the neutral axis passes through point *B*.
b) Determine the fully plastic moment for the vertical surface of loads.

Solution

a) The neutral axis bisects edge AC, $\alpha = 56.3°$, the centroid of *ABD* is $\left(\frac{20}{3}, -10\right)$ and of *BCD* $\left(-\frac{20}{3}, 10\right)$, $\varphi = 123.7°, d = 24.4$ cm, $M_p = 4.039$ kNm

b) The neutral axis bisects edge *BC*, $\alpha = -36.87°$, the centroid *ABD* is $\left(\frac{80}{3}, -10\right)$ and of *ACD* $\left(\frac{80}{3}, -30\right)$, $\varphi = 90°, d = 20$ cm, $M_p = 3.811$ kNm



For other angles of the loads surface a numerical procedure is needed.

Thanks for attention!