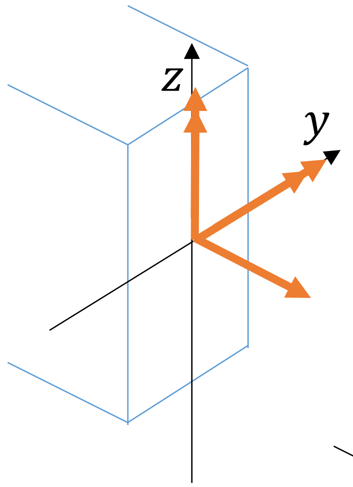
Strength of Materials

4. Composed bending, eccentric tension, cross-section core

Composed bending

(y, z) – principal central inertia axes

cross-sectional forces: N + My + Mz



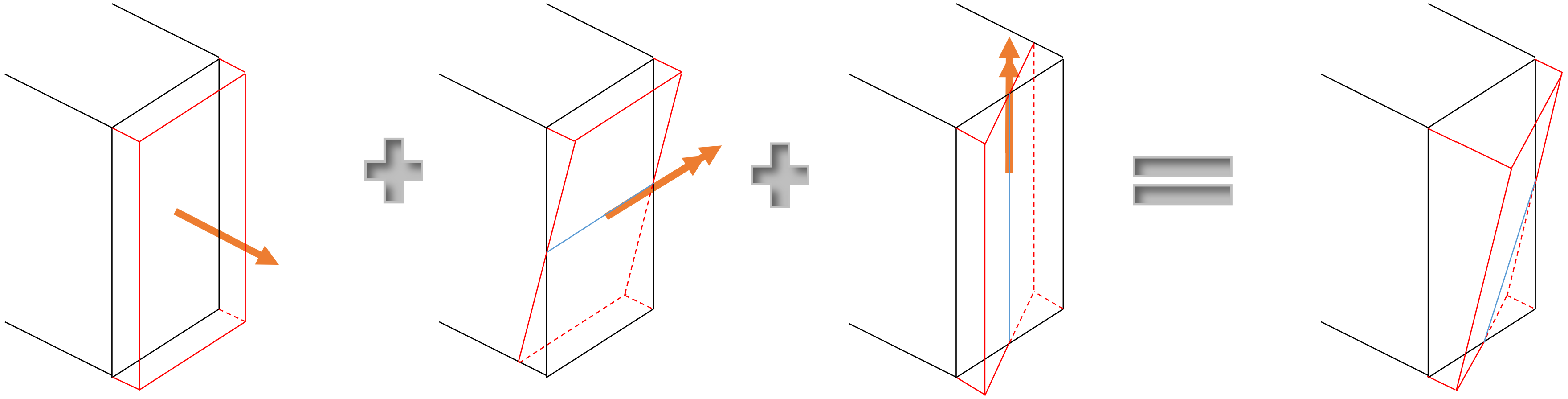
stress distribution: $\sigma_x = \frac{N}{A} + \frac{M_y}{I_y} z - \frac{M_z}{I_z} y$ surface equation

stress solid is limited by two surfaces: the cross-section surface and stress surface

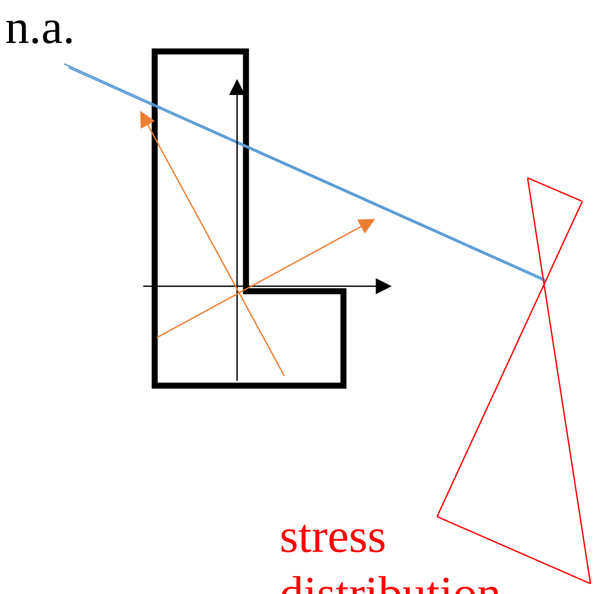
these surfaces intersect at the neutral axis, where $\sigma_x = 0$

$z = -\frac{N}{A}\frac{I_y}{M_y} - \frac{M_z}{I_z}\frac{I_y}{M_y} y$, the neutral axis is a straight line

it doesn't pass through the cross-section centroid

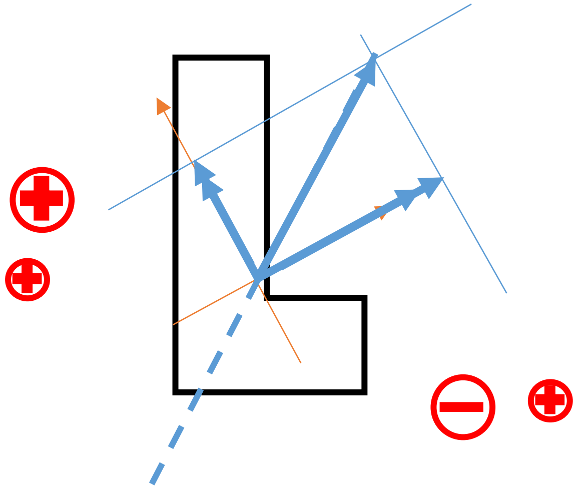


Composed bending – limit stress state



where is the maximum value of stress?
at the points the furthest from the neutral axis, of course!

does any method exist to check the calculated position of the neutral axis?



usually the influence of axial force is not predominant

Composed bending – limit stress state cont.

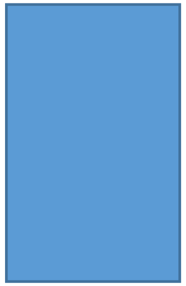
P – the most distant point from the neutral axis
$$\sigma_x = \frac{N}{A} + \frac{M_y}{I_y} z_P - \frac{M_z}{I_z} y_P, \qquad \max|\sigma_x| \le R$$

Calculations efficiency

search the most distant point or just check the stress at some points?

Problem:

A rectangular cross-section 2a × $a, A = 2a^2, I_y = \frac{2}{3}a^4, I_z = \frac{1}{6}a^4, y_p = \frac{a}{2}, z_p = a$



$$\sigma_x = \frac{N}{2a^2} + \frac{3M_y}{2a^3} - \frac{3M_z}{a^3} \le R$$

The unknown parameter occurs several times in the above formula

solve the cubical equation or use trial and error method ?

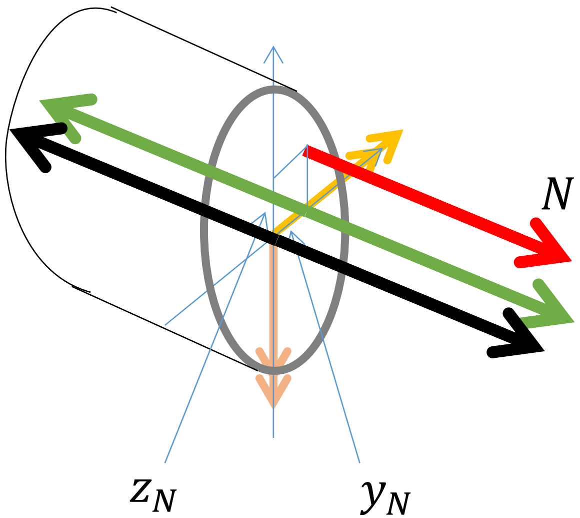
analytically: Cardano’s formulae; on computer: fast plot of the function for roots assessment

on calculator: break the formula up, next trial and error

$$\frac{N}{2a^2} \le R \to a_1 \qquad \frac{3M_y}{2a^3} \le R \to a_2 \qquad \left|-\frac{3M_z}{a^3}\right| \le R \to a_3 \qquad \to a \approx \max(a_1, a_2, a_3) + \delta$$

Keep in mind: the (very) precise values are not needed !

Eccentric tension - definition



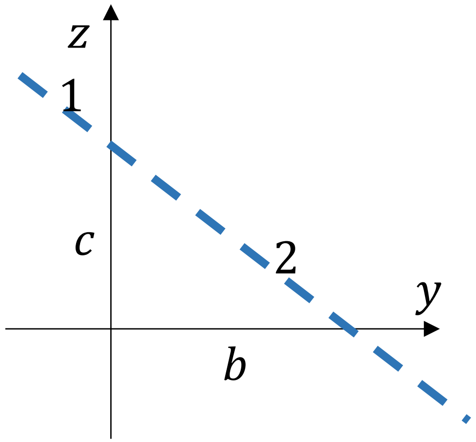
$$M_y = N z_N \qquad M_z = -N y_N \qquad N$$

$$\sigma_x = \frac{N}{A} + \frac{M_y}{I_y} z - \frac{M_z}{I_z} y = \frac{N}{A} + \frac{N z_N}{A i_y^2} z + \frac{N y_N}{A i_z^2} y = \frac{N}{A}\left(1 + \frac{z_N z}{i_y^2} + \frac{y_N y}{i_z^2}\right)$$

neutral axis: $\sigma_x = 0 \rightarrow 1 + \frac{z_N z}{i_y^2} + \frac{y_N y}{i_z^2} = 0$ not depends on the force value

$$b \stackrel{\text{def}}{=} -\frac{i_z^2}{y_N}, c \stackrel{\text{def}}{=} -\frac{i_y^2}{z_N} \rightarrow \frac{y}{b} + \frac{z}{c} = 1$$

the intercept form of the neutral axis



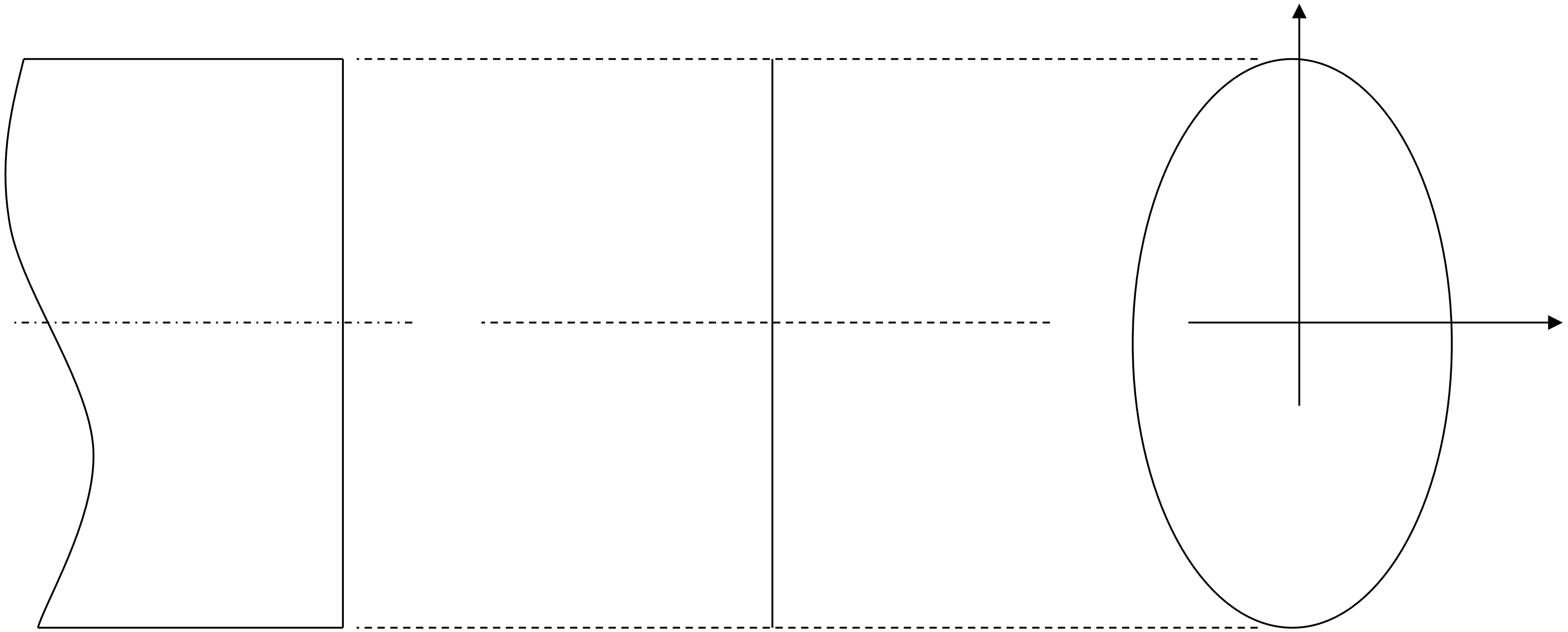
a line through the points 1 and 2

$$y = (y_2 - y_1)t + y_1$$
$$z = (z_2 - z_1)t + z_1$$

$$z = 0 \rightarrow t = -\frac{z_1}{z_2 - z_1} \rightarrow y = b = -\frac{y_2 - y_1}{z_2 - z_1} z_1 + y_1$$

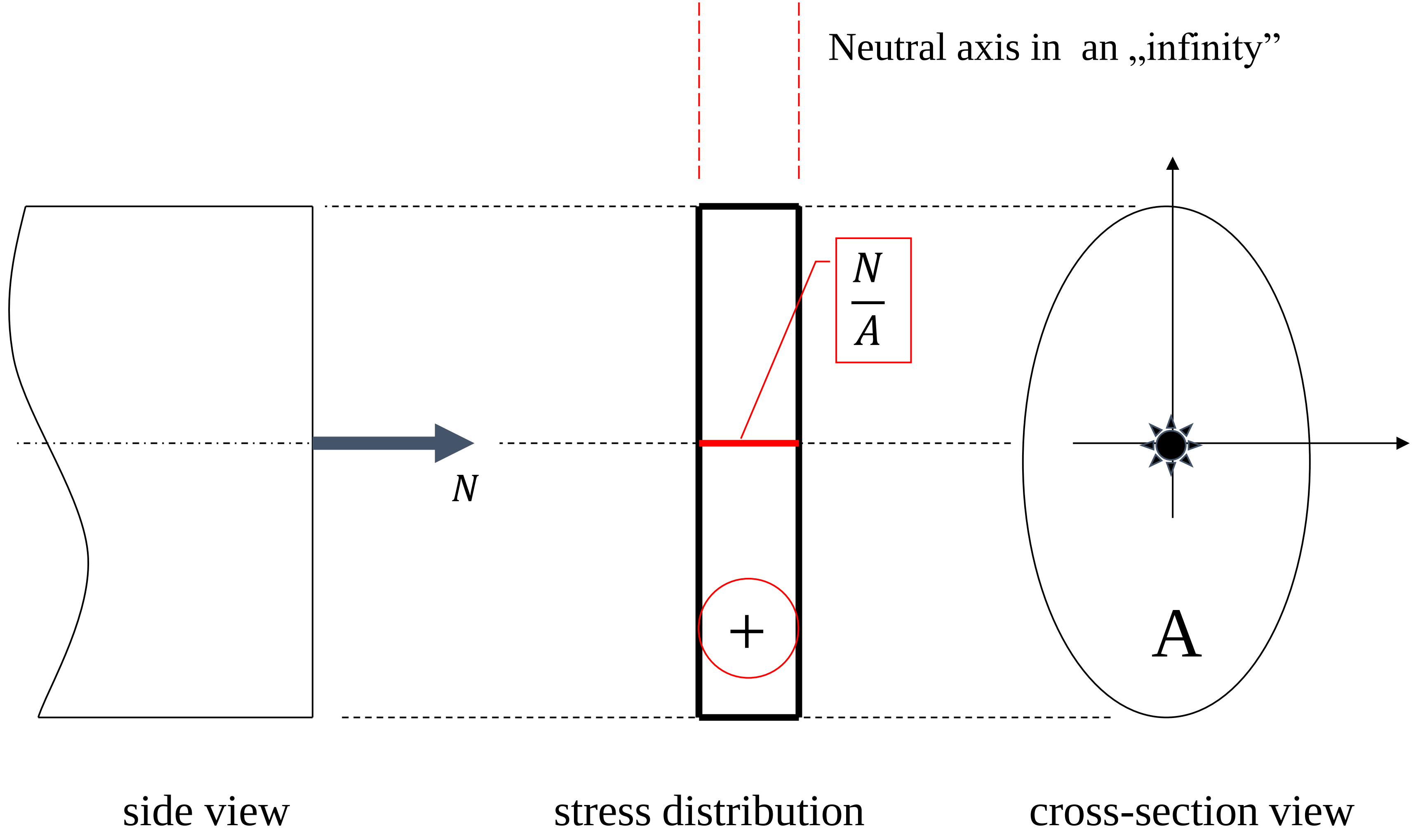
$$y = 0 \rightarrow t = -\frac{y_1}{y_2 - y_1} \rightarrow z = c = -\frac{z_2 - z_1}{y_2 - y_1} y_1 + z_1$$

Eccentric tension – neutral axis position

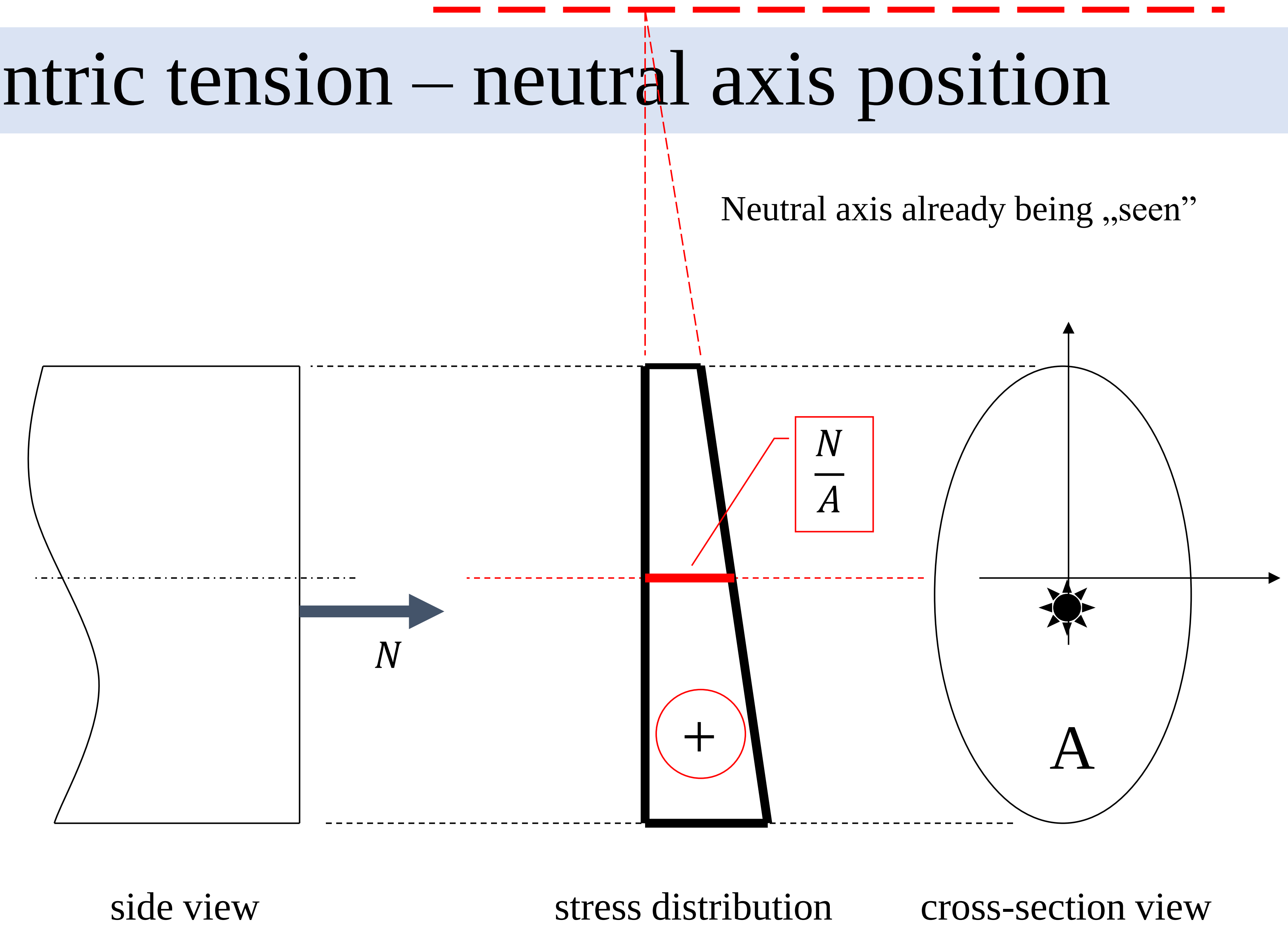


side view stress distribution cross-section view

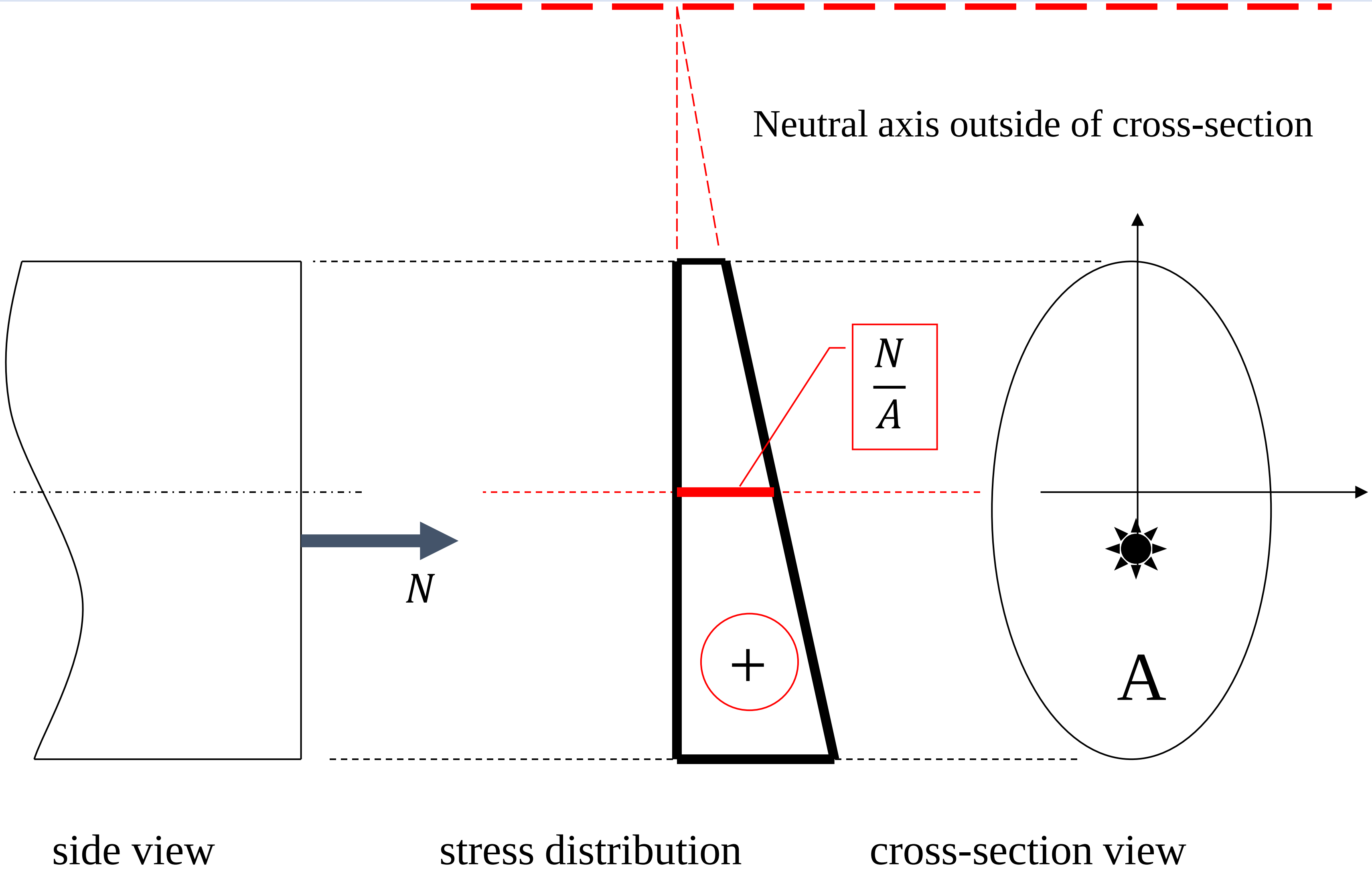
Eccentric tension – neutral axis position



Eccentric tension – neutral axis position

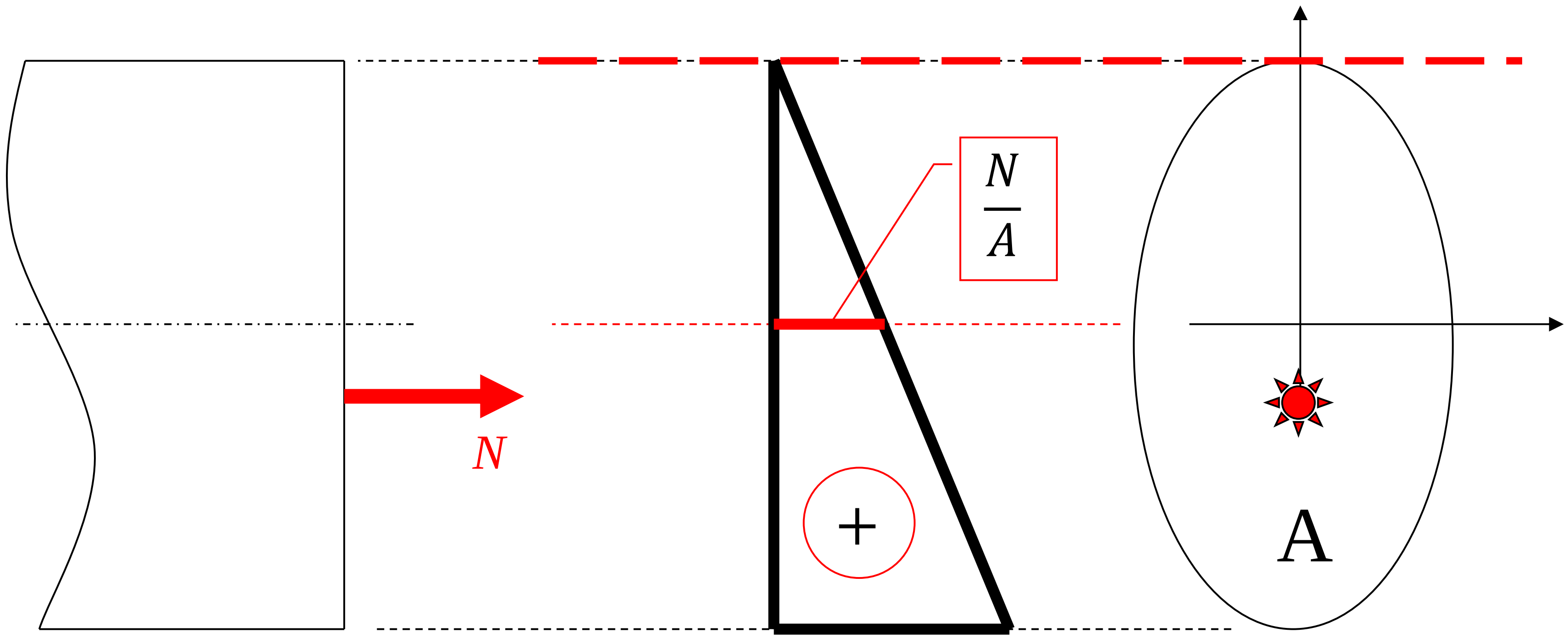


Eccentric tension – neutral axis position



Eccentric tension – neutral axis position

Neutral axis touching cross-section contour

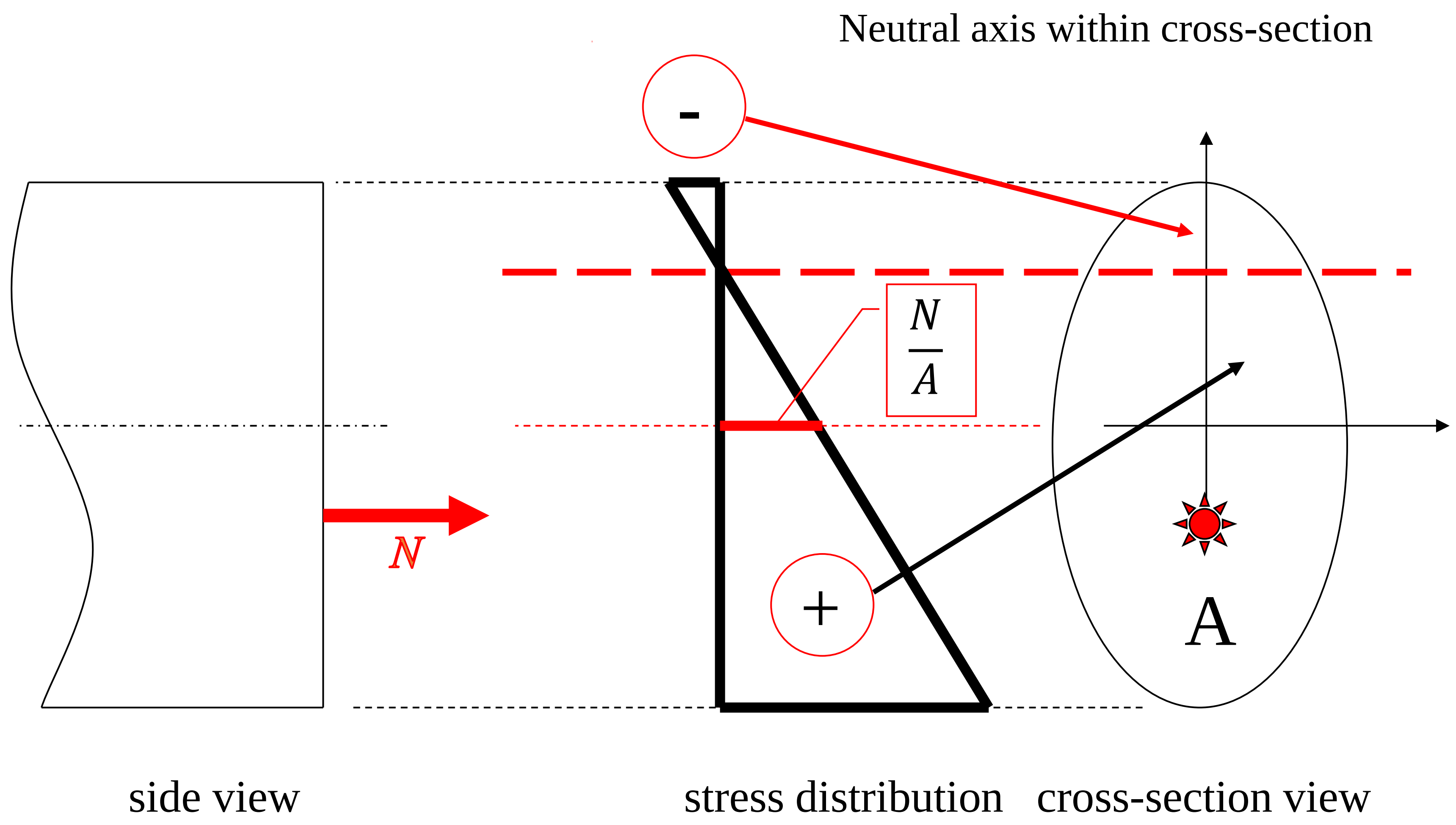


side view

stress distribution

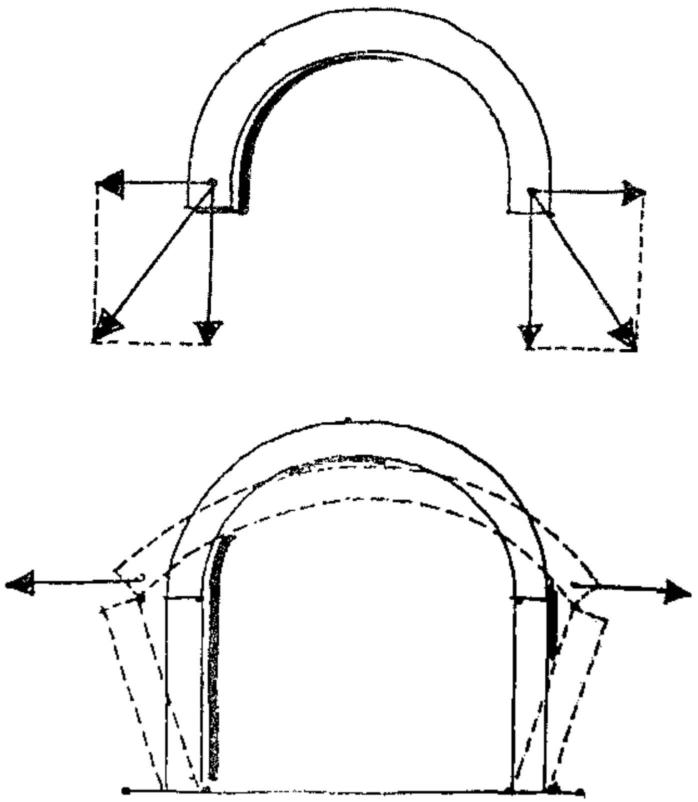
cross-section view

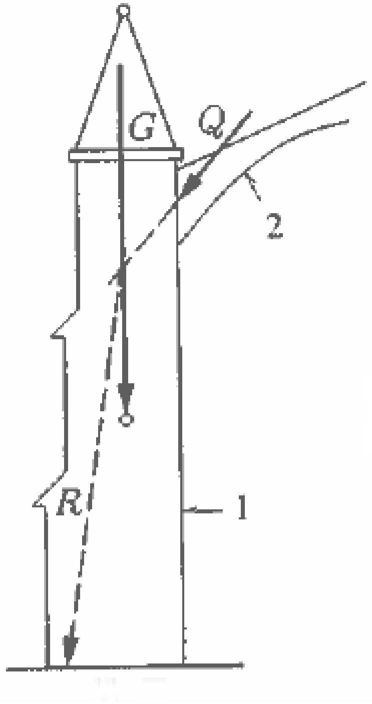
Eccentric tension – neutral axis position

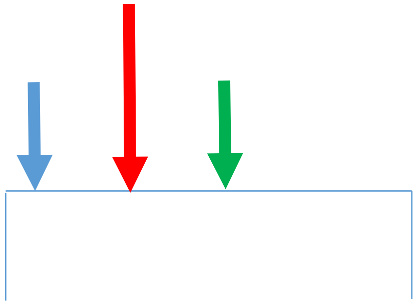


Eccentric tension – resultant force position

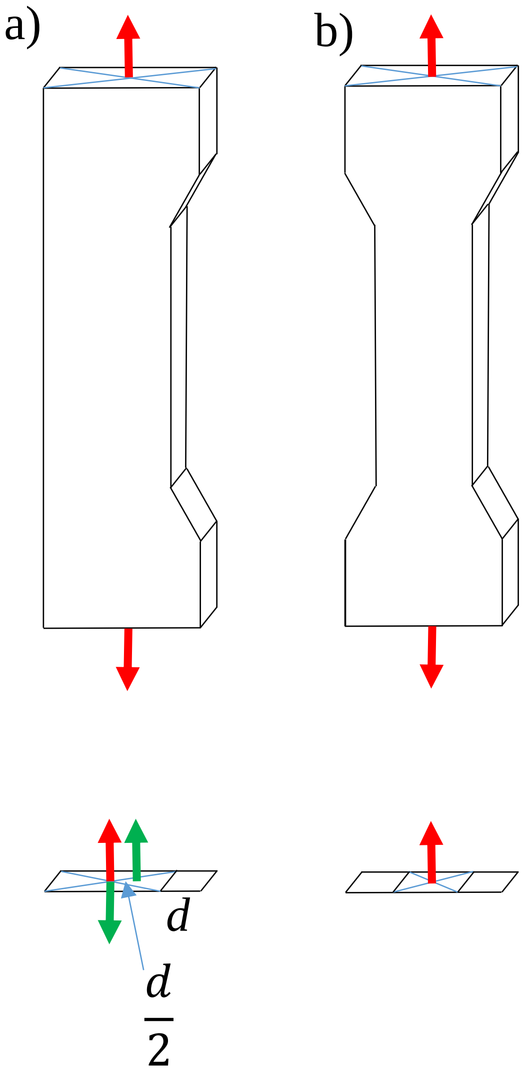
medieval cathedrals – aisle height:
Wien (Austria) – 22.4 m
Burgos (Spain) – 25 m
Mariacki (Cracow) – 28 m
York Minster (York) – 31 m
Notre Dame (Paris) – 32.5 m
Notre Dame (Chartres) – 36.55 m
Notre Dame (Reims) – 38 m
St. Peter & SM (Koln) – 43.35 m
Santa Maria (Milan) – 45 m
Santa Maria di Fiore – 43 m
(70 m with dome)







Eccentric tension - example



cut-out: a) from one side, b) from both sides

dimensions: width $h = 10$ mm, thickness $b = 1$ mm, cut-out width $d = 3$ mm

1) calculate $\max\sigma_x$ if $N = 1$ kN

2) determine the bearing capacity if $R = 350$ MPa

Solution

1) normal stress in the middle cross-section

a) $\sigma_x = \frac{N}{b(h-d)} + \frac{N\cdot\frac{d}{2}\cdot 12}{b(h-d)^3}\cdot\frac{h-d}{2} = \frac{1\cdot 10^3}{0.001\cdot 0.007} + \frac{1\cdot 10^3\cdot 0.0015\cdot 12}{0.001\cdot 0.007^3}\cdot 0.0035 = 327\text{ MPa}$

b) $\sigma_x = \frac{N}{b\cdot(h-d)} = \frac{1\cdot 10^3}{0.001\cdot 0.004} = 250\text{ MPa}$

2) bearing capacity

a) $N = \frac{R}{\frac{1}{b(h-d)} + \frac{6d}{b(h-d)^3}\cdot\frac{h-d}{2}} = \frac{350\cdot 10^6}{\frac{1}{0.001\cdot 0.007} + \frac{6\cdot 0.003}{0.001\cdot 0.007^3}\cdot 0.0035} = 1.07\text{ kN}$

b) $N = R\cdot b\cdot(h-d) = 350\cdot 10^6\cdot 0.001\cdot 0.004 = 1.4\text{ kN}$

Conclusion: the case b) is more favorable

Eccentric tension – neutral axis position

$$1 + \frac{zz_N}{i_y^2} + \frac{yy_N}{i_z^2} = 0$$

Dual interpretation of neutral axis equation

(y, z) – regards the neutral axis
(y_N, z_N) – regards the acting force

if the point of the force position is known – we get the neutral axis equation

point (force) → line (n.a.)

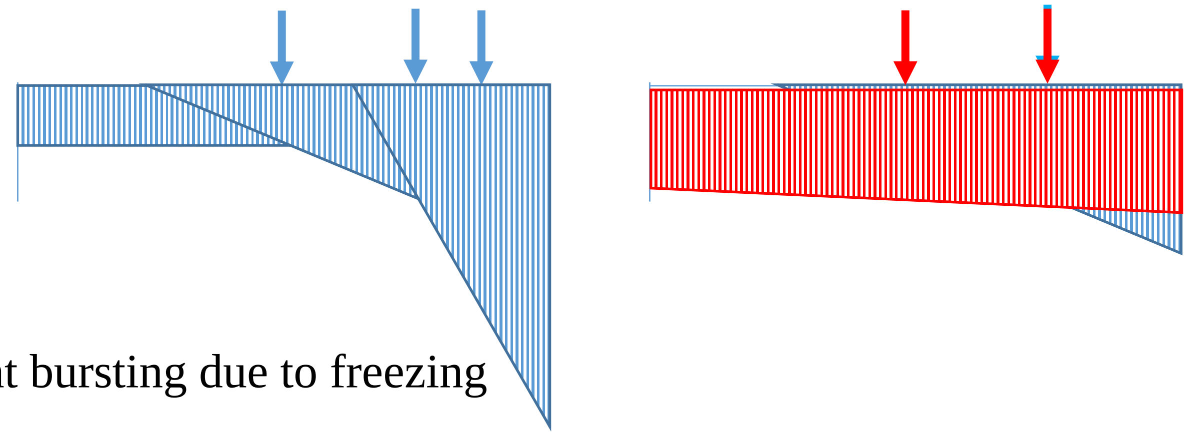
if the point of the neutral axis is known – we get the line of the applied force action

point (n.a.) → line (force)

Eccentric tension – cross-section core

ceramic materials:
very good strength in compression
(breaking height in compression:
steel 5.6 km, granite over 8 km)
very poor strength in tension, so:

unidirectional material behavior



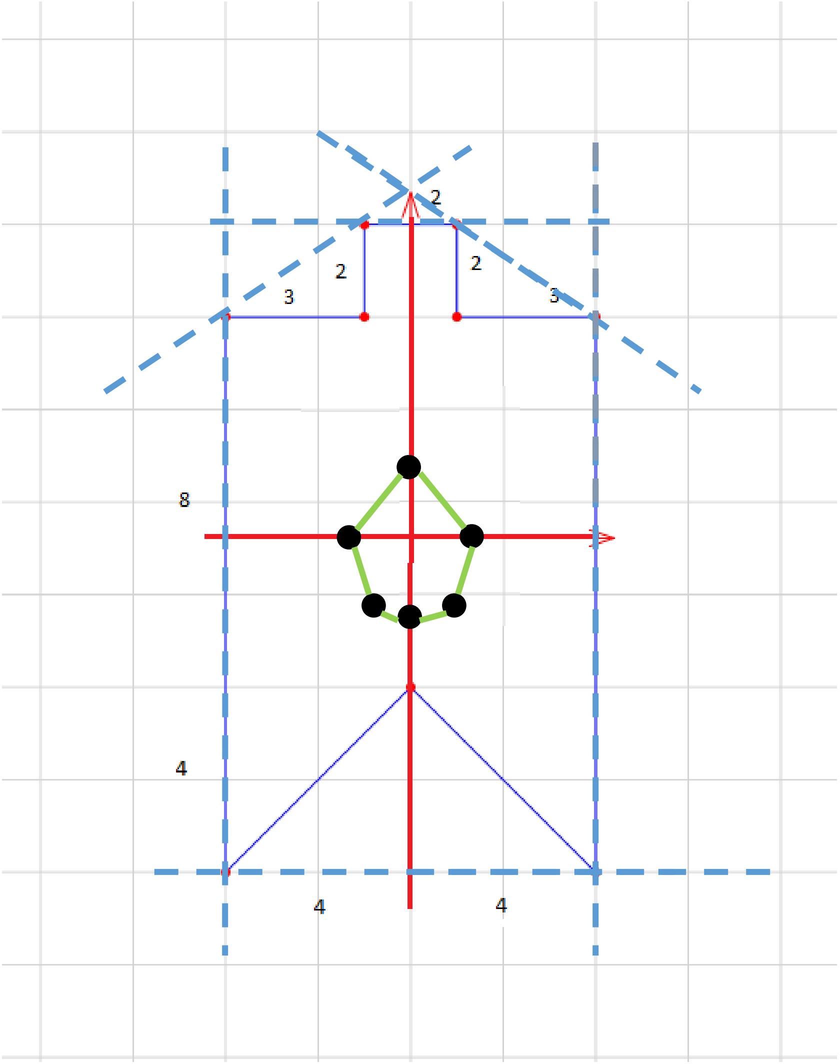
the part of the cross-section in tension is unused
usually this part is visibly cracked
it is susceptible to humidity intrusion and subsequent bursting due to freezing
soil under footing is not consolidated

it is highly desirable the whole cross-section works; the problem is: where the force should be applied to avoid tension stress
this happen when the neutral axis will be outside of the cross-section or at least touching the cross-section outlined (the cross-section outline: the smallest convex figure containing entire cross-section)

a cross-section core: a locus of the eccentric force position causing a stress of one sign in the cross-section

The cross-section core is limited from without by a core curve and corresponds to the all neutral axes touching the cross-section (cross-section outlined)

Cross-section core - example



calculations

cross-section characteristics: $A = 84, C(4.0, 7.222),$

$I_1 = 861.19, I_2 = 470.67, i_y^2 = 10.25, i_z^2 = 5.603$

given $(b, c) \rightarrow y_N = -\frac{i_z^2}{b}, z_N = -\frac{i_y^2}{c}$

line $z = 6.778$:

$$b \rightarrow \infty, y_N = 0, c = 6.778, z_N = -1.512$$

line $1(1, 6.778),\ 2(4, 4.778)$

$$y = (4-1)t + 1, y = 0 \rightarrow t = -\frac{1}{3} = -0.3333,$$

$$z = (-2)(-0.3333) + 6.778 = 7.445 = c, z_N = -\frac{10.25}{7.445} = -1.377$$

$$z = 0 \rightarrow t = 3.389 \rightarrow y = 3 \cdot 3.389 + 1 = 11.167 = b \rightarrow y_N = -0.5717$$

line $y = 4$:

$$c \rightarrow \infty, z_N = 0, b = 4, y_N = -1.401$$

line $z = -7.222$:

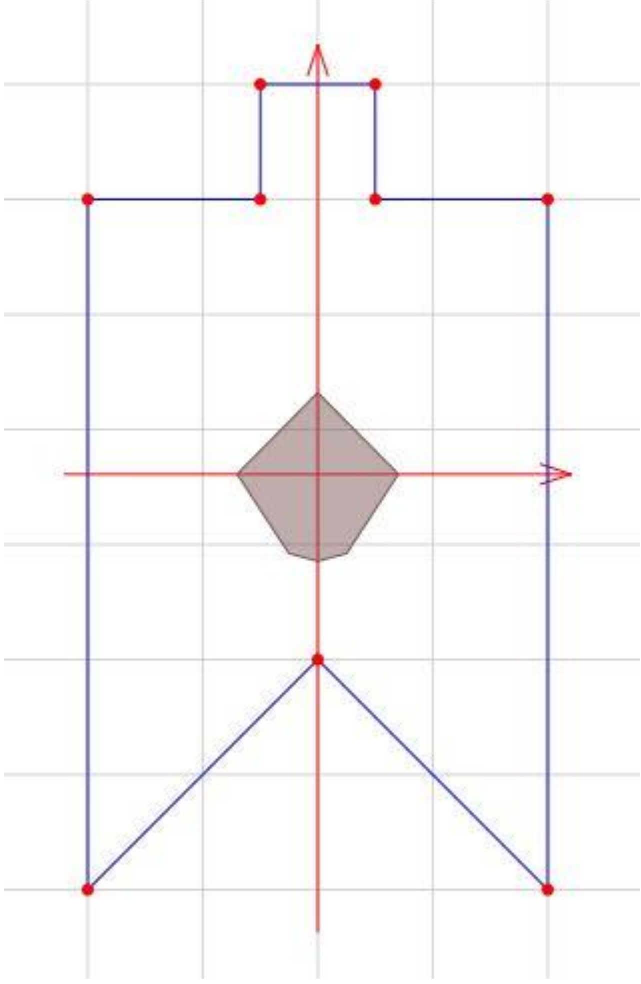
$$b \rightarrow \infty, y_N = 0, c = -7.222, z_N = 1.419$$

Cross-section core – example cont.

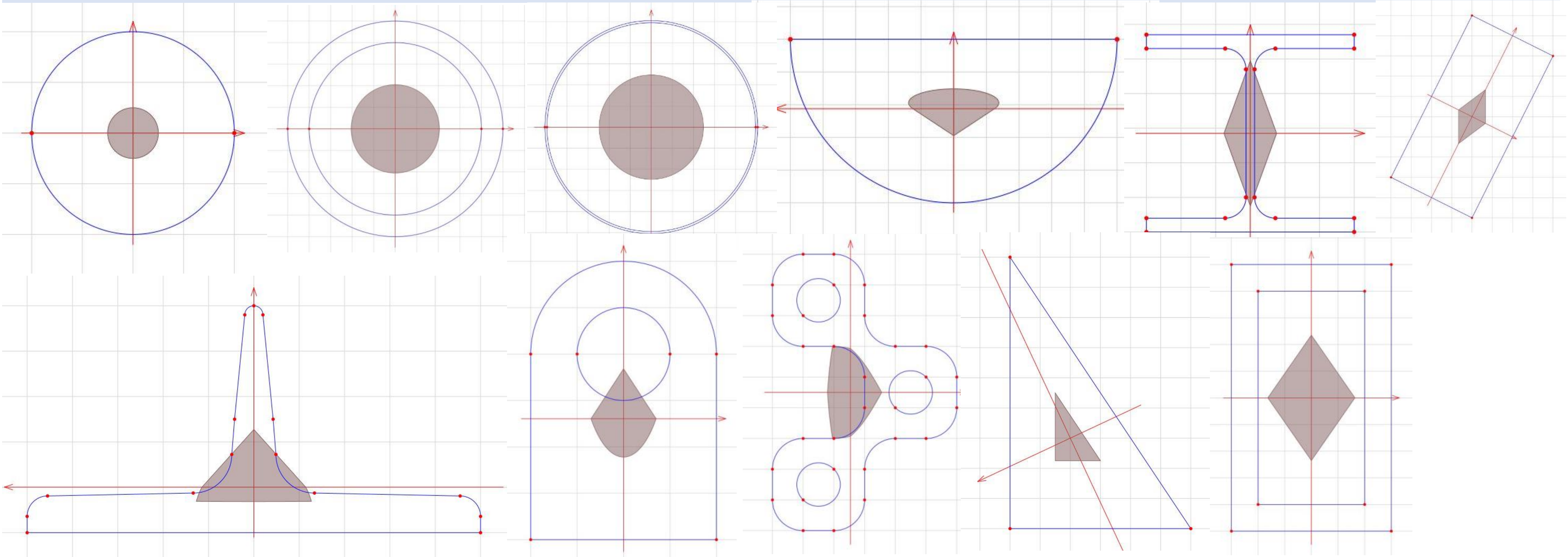
The solution from the famous program „section”:

```
Pole powierzchni: 84.0
Środek ciężkości: (4.0, 7.2222)
Centralne momenty bezwładności: Iy=861.19, Iz=470.67, Iyz=0.0
Główne centralne momenty bezwładności: I₁=861.19, I₂=470.67
Kąt głównych centralnych momentów bezwładności: 0.0 rad, 0.0°
Macierz przekształcenia do współrzędnych głównych centralnych:
       1.0,         0.0,       -4.0
       0.0,         1.0,    -7.2222
         0,           0,          1
Wskaźniki wytrzymałości na zginanie (sprężyste): W₁=119.24, W₂=117.67

Punkty rdzenia przekroju (współrzędne wyjściowe / główne centralne):
(4.0, 8.6418)  /  (0.0, 1.4195)
(2.5992, 7.2222)  /  (-1.4008, 0.0)
(3.4982, 5.8451)  /  (-0.50178, -1.3772)
(4.0, 5.7096)  /  (0.0, -1.5126)
(4.5018, 5.8451)  /  (0.50178, -1.3772)
(5.4008, 7.2222)  /  (1.4008, 0.0)
```



Cross-section core - examples



the cross-section core is always convex
it surrounds the centroid
when the cross-section is n-polygon, the core is an n-polygon also
never exceeds the cross-section outline
the core is a purely geometric feature of the cross-section

That’s all, folks!