Strength of Materials

5. Transverse bending

Transverse bending - definition

simple bending is relatively rare loading condition – it implies constant value of bending moment
usually, the bending moment is not constant, we call that case:

- transverse bending loads
- transverse bending
- non-uniform bending

we know from the statics course, that

$$M \neq \text{const} \rightarrow Q \neq 0$$

it means, that there are two cross-sectional forces in the same time: bending moment and shear force
we have to consider simultaneous actions of these forces
action of bending moment were already discussed, but it is not clear the presence of shear force changes or modifies the previous results
there are:

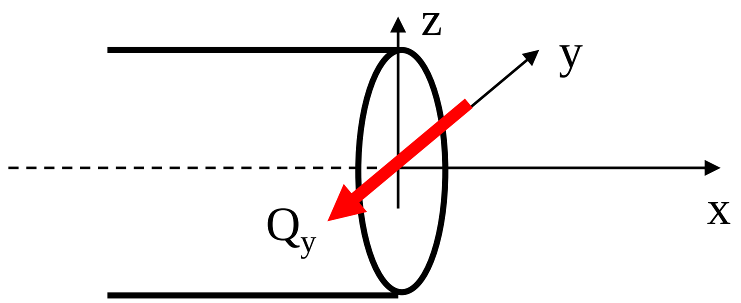
- pure shear
- shear(ing) (in general)

Shear - definition

the set of cross-sectional forces reduces solely to the shear force vector perpendicular to the bar axis

$M_x = 0$, $M_y = 0$, $M_z = 0$

$Q_x = N = 0$, $Q_y \neq 0$, $Q_z = 0$

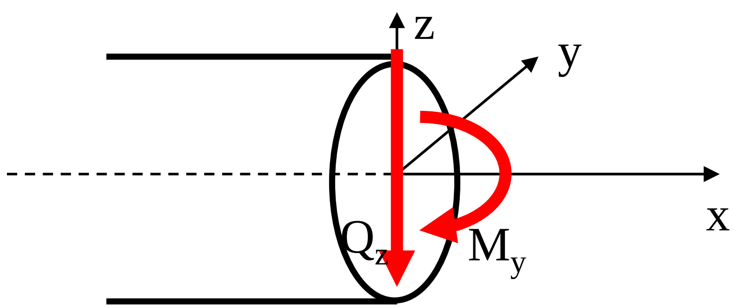


or

$M_x = 0$, $M_y \neq 0$, $M_z = 0$

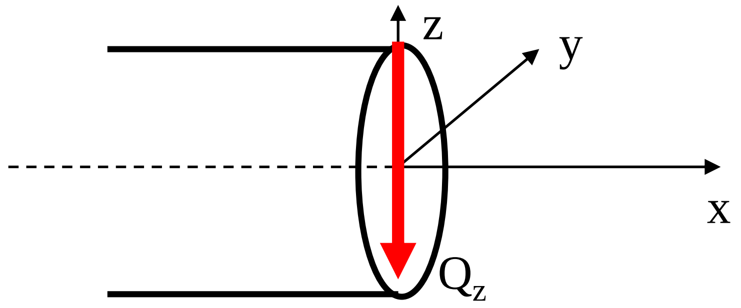
$Q_x = N = 0$, $Q_y = 0$, $Q_z \neq 0$

Shear is associated with bending !



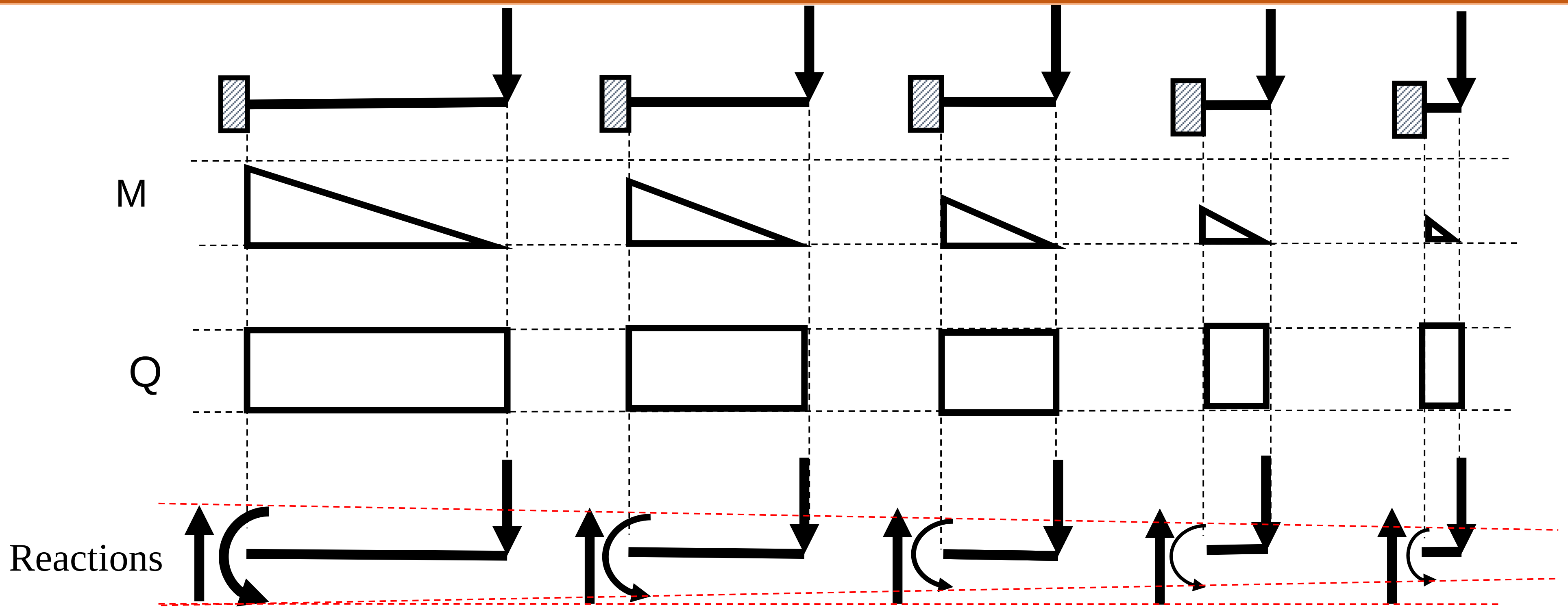
NON-UNIFORM BENDING

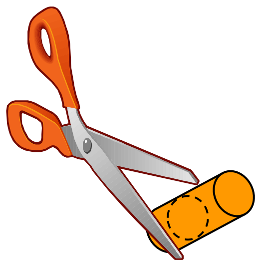
$Q_x = N = 0$, $Q_y = 0$, $Q_z \neq 0$



PURE SHEAR

Shear – pure shear explication





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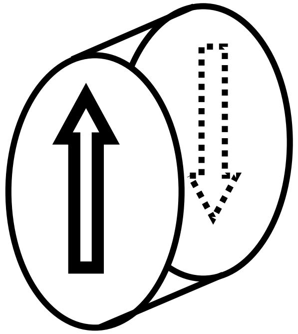
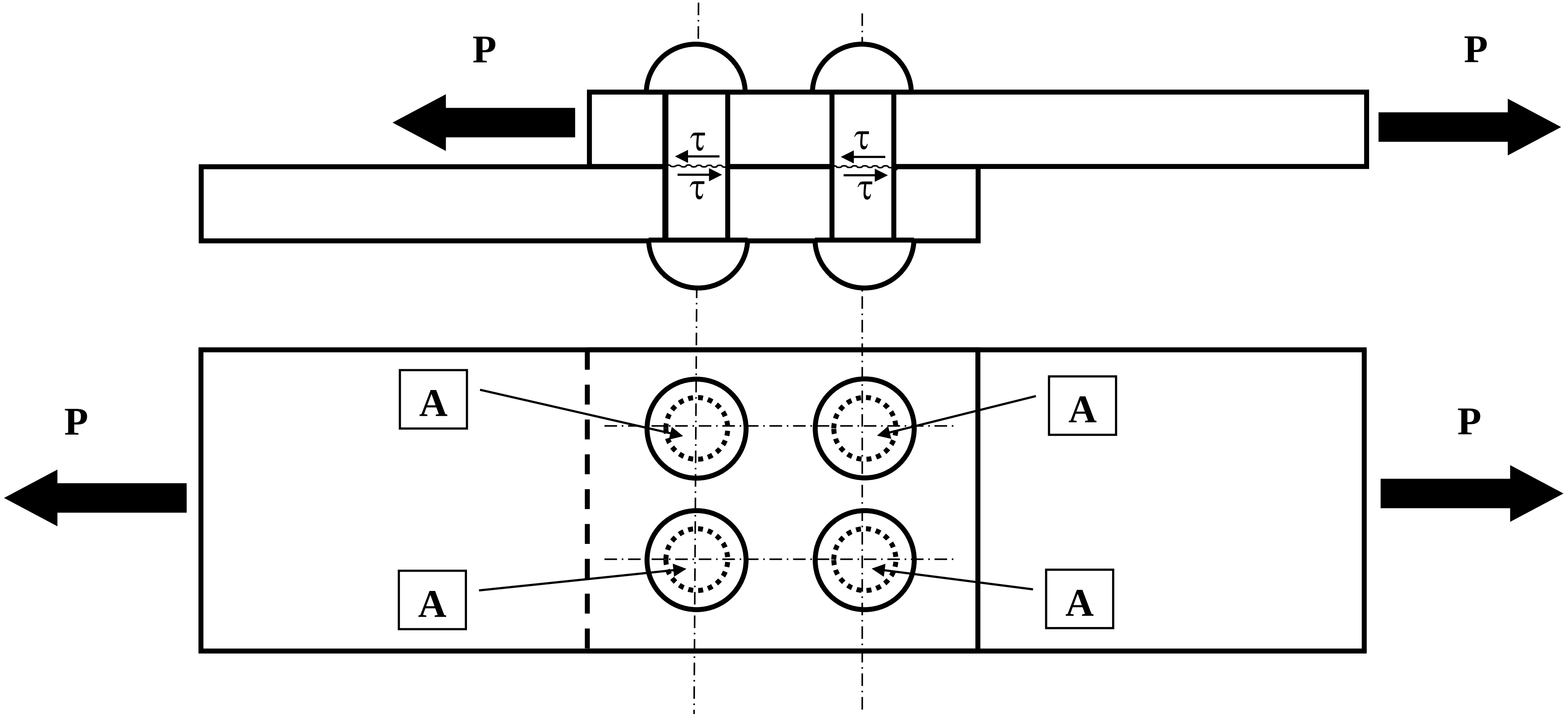


$M_u \to 0$

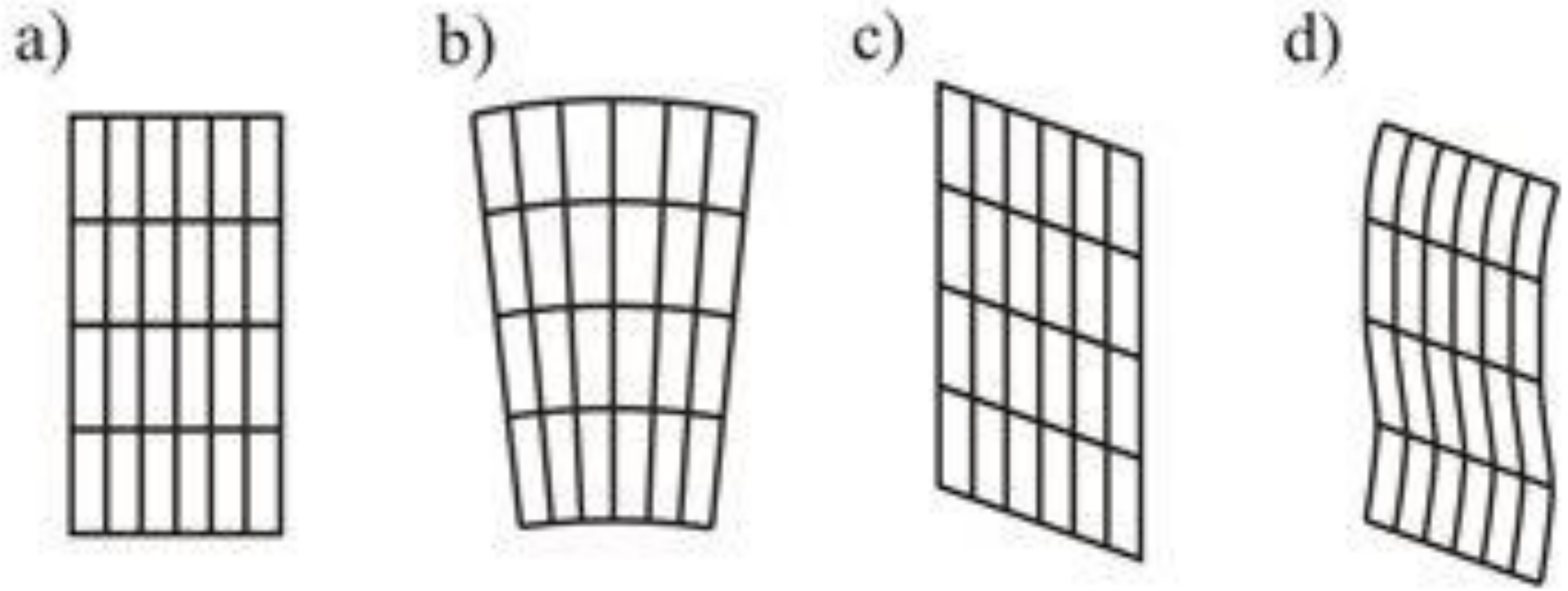
$Q = \mathrm{const}$

Pure shear



$4A\tau = P$ → τ = P/4A mean shear stress

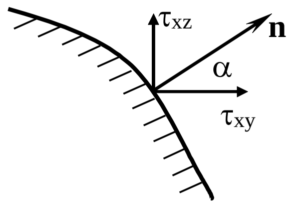
Transverse bending – cross-section warping



a) before loading,
b) circular bending (all angles remain right, no shear strains and stress),
c) all angles alike (constant shear strains and stress, not allowed at the top/bottom)
d) right angles at top/bottom, shear strain and stress between

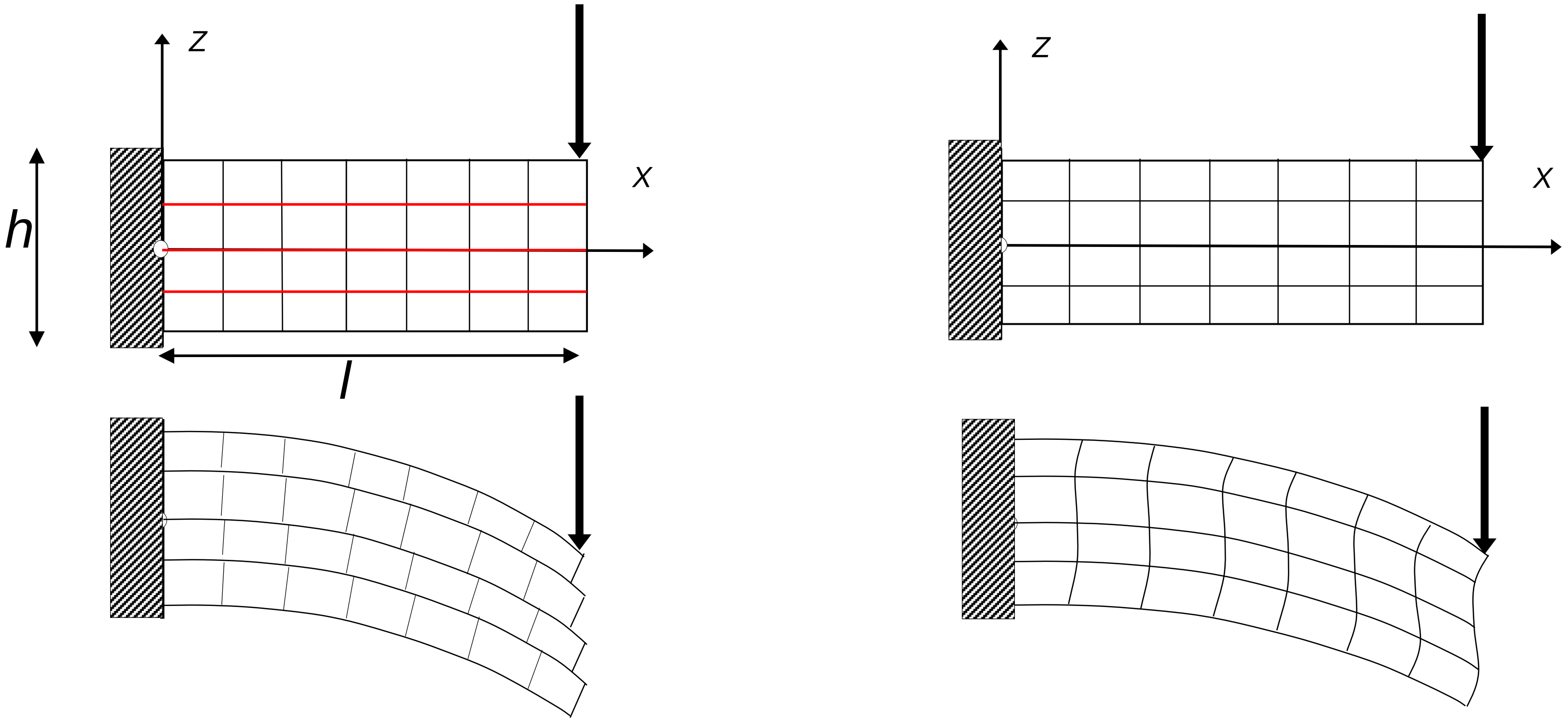
$$\mathbf{n}(0, \cos\alpha, \sin\alpha), \quad q_{nx} = 0$$

$$\Rightarrow \quad \tau_{xy} \cos\alpha + \tau_{xz} \sin\alpha = 0$$



it can be proved that on the free boundary stress vector should be tangent to the boundary;
no shear stress perpendicular to the boundary allowed

Transverse bending – cross-section warping

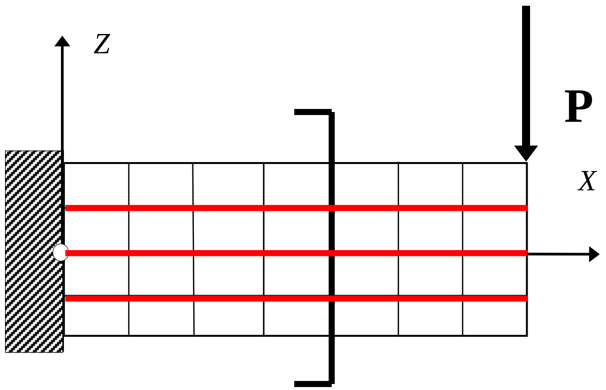


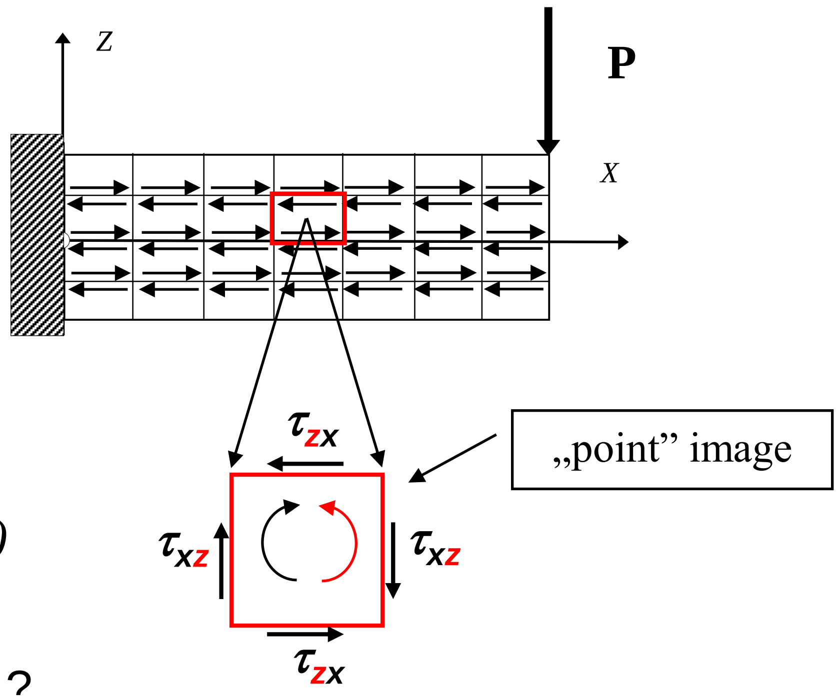
For $h/l \ll 1$ distortion is small and we will use the formula for normal stress derived from this assumption :

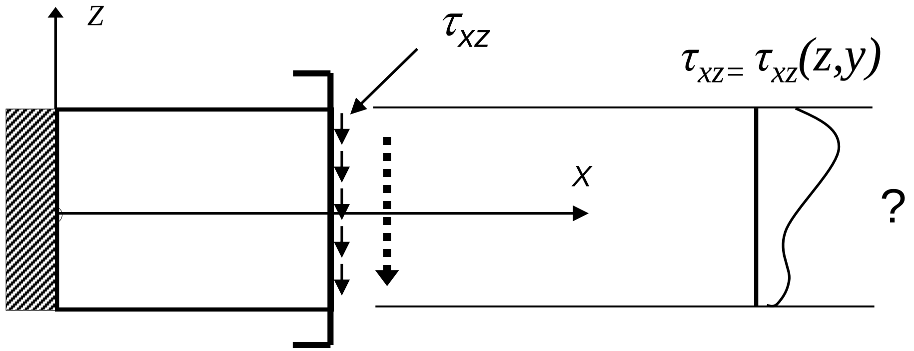
$$\sigma_x = \frac{M_y}{J_y} z$$

Bernoulli hypothesis as well as symmetry conditions of plane cross-sections do not hold!!

Transverse bending – shear stress

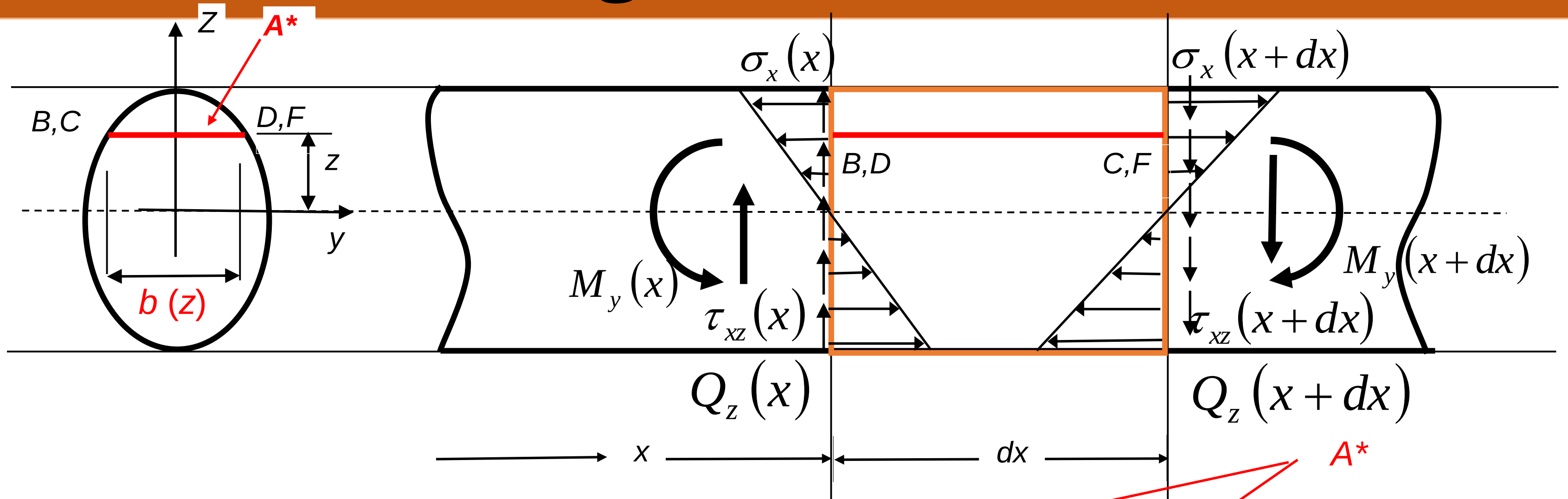




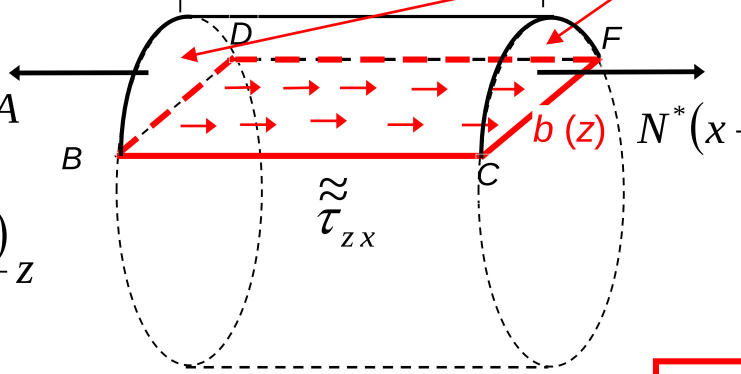


$$Q_z = P = \int_A \tau_{xz} dA$$

Transverse bending – mean shear stress



Prismatic bar!



$$N^*(x) = \int_{A^*} \sigma_x(x)\,dA \qquad N^*(x+dx) = \int_{A^*} \sigma_x(x+dx)\,dA$$

$$\sigma_x = \frac{M_y}{J_y} z \qquad \sigma_x(x+dx) = \frac{M_y(x+dx)}{J_y} z$$

$$\lim_{dx\to 0}\left[N^*(x) = N^*(x+dx) + \tilde{\tilde{\tau}}_{zx} \cdot b(z) \cdot dx\right] \longrightarrow \boxed{\tilde{\tau}_{zx} = \frac{Q_z(x) \cdot S_y^*(z)}{I_y \cdot b(z)} = \tilde{\tau}_{xz}}$$

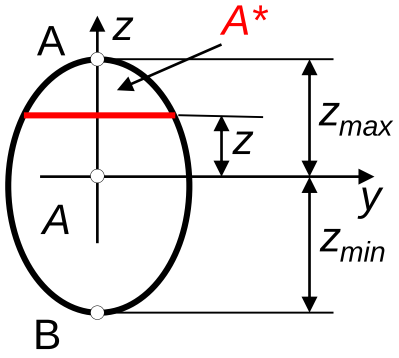
Transverse bending – shear stress distribution

Formula $\tilde{\tau}_{xz} = \frac{1}{I_y}[Q_z(x)]\left[\frac{S^*(z)}{b(z)}\right]$ holds for prismatic bars only!

and is given in principal central axes of cross-section inertia.

$$\max \tilde{\tau}_{xz} = \left(\frac{1}{I_y}\right)\max\big|_x [Q_z(x)] \cdot \max\big|_z \left[\frac{S^*(z)}{b(z)}\right]$$

Distribution along z-axis



For A: $S^*(z_{max})=0$ since $A^*=0$ $\Longrightarrow$ $\tilde{\tau}_{xz} = 0$

For B: $S^*(z_{min})=0$ since $A^*=A$ $\Longrightarrow$ $\tilde{\tau}_{xz} = 0$

Also:

Static Boundary Conditions in A and B:

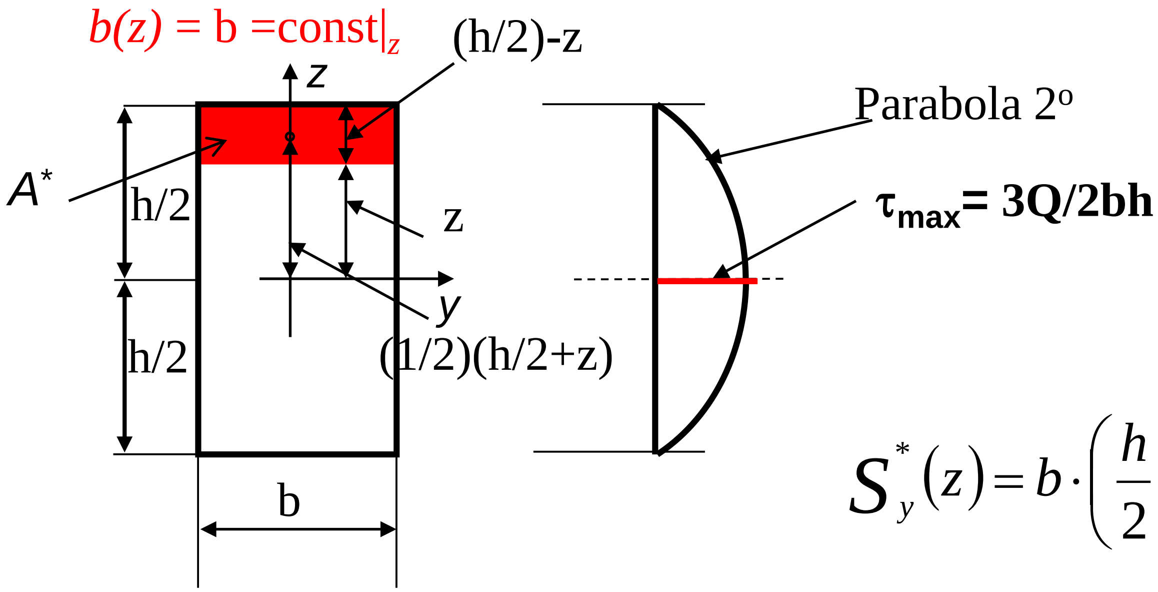
$$q_i = \sigma_{ij} n_j$$

$$q_x = 0 = \sigma_x \cdot 0 + \tau_{xy} \cdot 0 + \hat{\tau}_{xz} \cdot 1 \rightarrow \hat{\tau}_{xz} = 0$$

$$q_y = 0 = \tau_{xy} \cdot 0 + \sigma_y \cdot 0 + \tau_{yz} \cdot 1 \rightarrow \tau_{yz} = 0$$

$$q_z \neq 0 = \hat{\tau}_{xz} \cdot 0 + \tau_{yz} \cdot 0 + \sigma_z \cdot 1 \rightarrow \sigma_z = q_z$$

Transverse bending – stress distribution cont.



$$\tilde{\tau}_{zx} = \frac{Q_z(x)\cdot S_y^*(z)}{I_y \cdot b(z)} = \tilde{\tau}_{xz}$$

$$S_y^*(z) = b\cdot\left(\frac{h}{2} - z\right)\cdot\frac{1}{2}\left(\frac{h}{2} + z\right) = \frac{b}{2}\left(\frac{h^2}{4} - z^2\right)$$

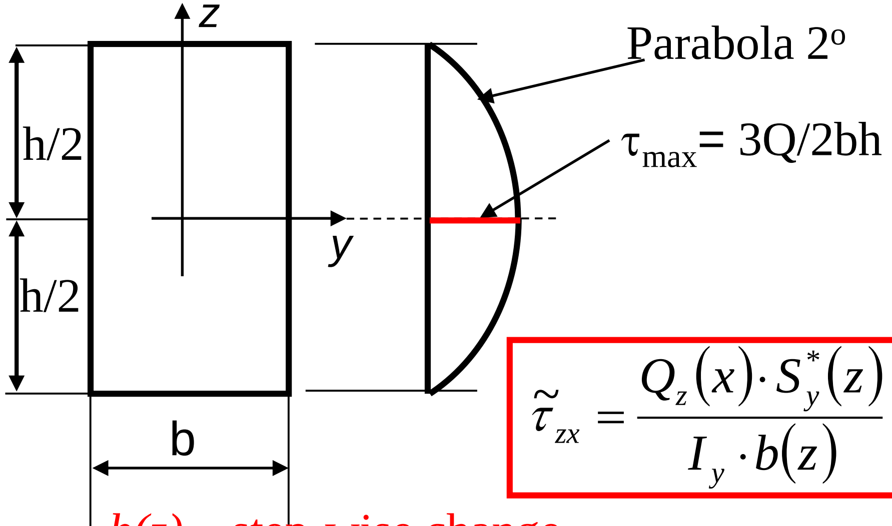
$$\tilde{\tau}_{zx} = Q_z(x)\frac{b/2}{(bh^3/12)\cdot b}\left(\frac{h^2}{4} - z^2\right) = \tilde{\tau}_{xz}$$

For $z = 0$

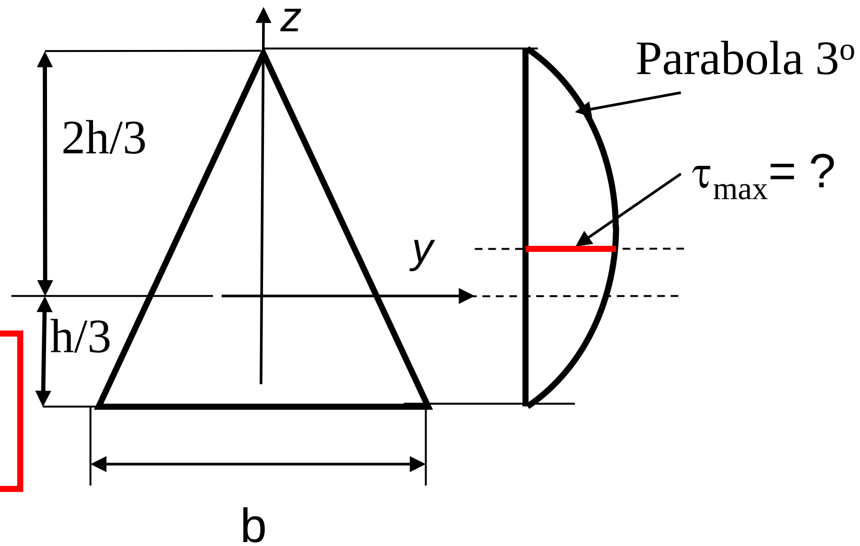
$$\tilde{\tau}_{zx} = Q_z(x)\frac{b/2}{(bh^3/12)\cdot b}\left(\frac{h^2}{4} - 0\right) = Q_z\frac{12bh^2}{8b^2h^3} = \frac{3Q_z}{2bh}$$

Mean shear stress – some results

$b(z) = b = \text{const}|_z$

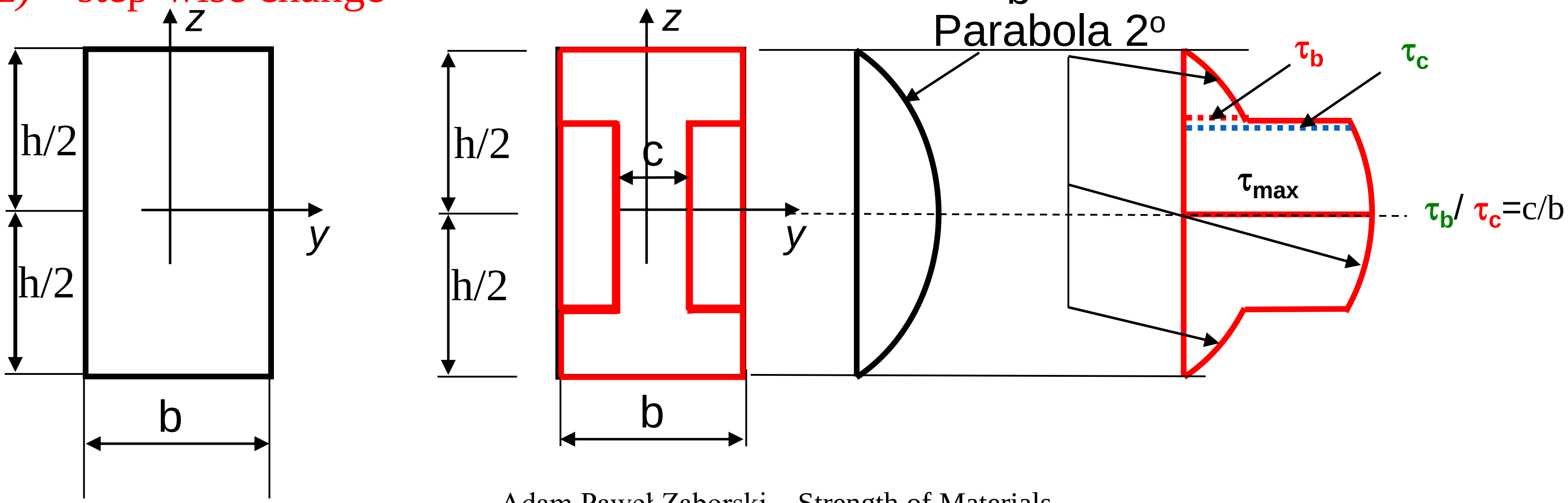


$b(z)$ – linear function of z



$$\tilde{\tau}_{zx} = \frac{Q_z(x)\cdot S_y^*(z)}{I_y \cdot b(z)} = \tilde{\tau}_{xz}$$

$b(z)$ – step-wise change



Transverse bending – stress trajectories

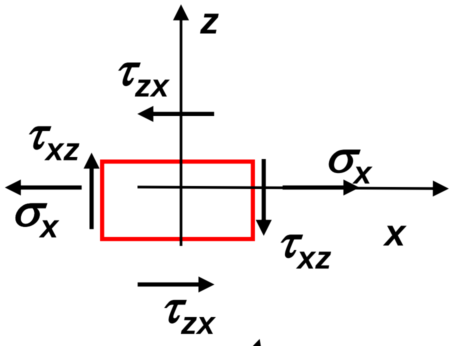
$$\sigma_x = \frac{M_y(x)}{J_y} z$$

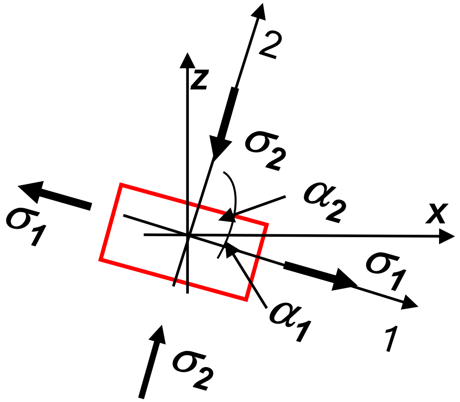
$$\tau_{zx} = \frac{Q_z(x) \cdot S_y^*(z)}{J_y \cdot b(z)} = \tau_{xz}$$

$$\sigma_{1,2} = \frac{\sigma_x}{2} \pm \frac{1}{2}\sqrt{\sigma_x^2 + 4\tau_{xz}^2}$$

$$tg\alpha_1 = \frac{\sigma_1 - \sigma_x}{\tau_{xz}} \qquad tg\alpha_1 = \frac{\tau_{xz}}{\sigma_1 - \sigma_z}$$

$$\boxed{\sigma_z = 0} \qquad tg\alpha_{1,2} = \frac{\tau_{xz}}{\sigma_{1,2}}$$

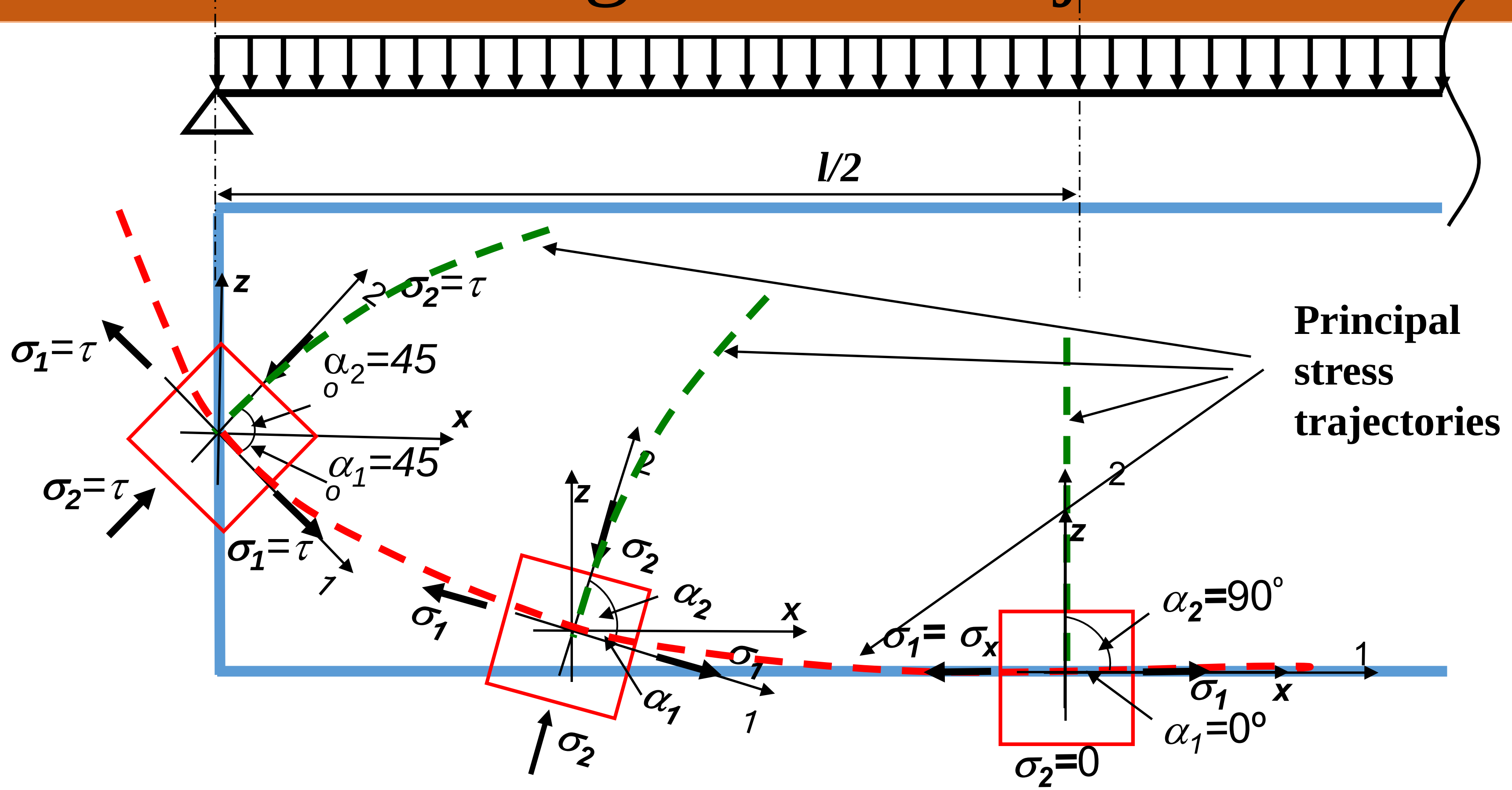




for z=0: $\sigma_x = 0 \Rightarrow \sigma_{1,2} = \tau_{xz}$

$tg\alpha_{1,2} = \mp 1 \Rightarrow \alpha_{1,2} = \mp 45^o$

Transverse bending – stress trajectories cont.

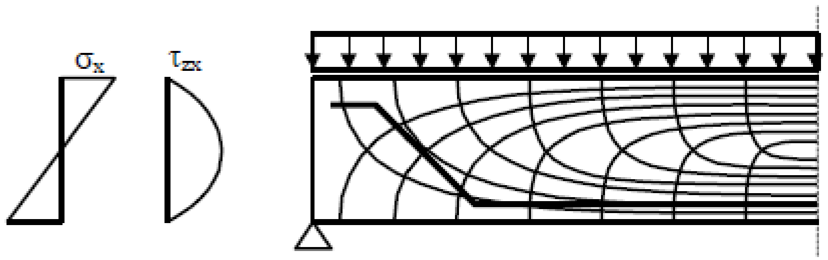


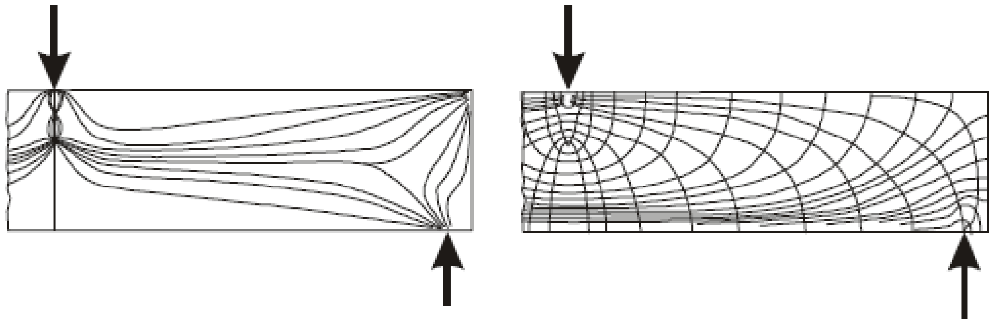
$M_y = 0$ $\quad Q_z \neq 0$

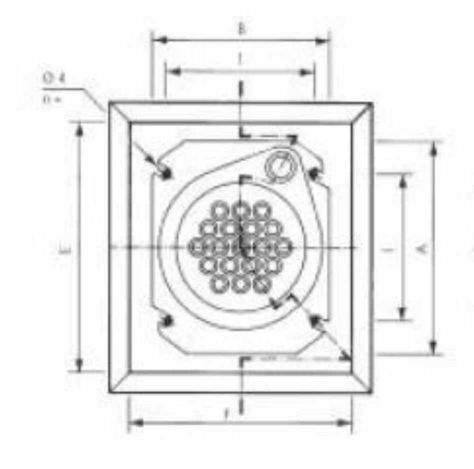
$M_y \neq 0$ $\quad Q_z \neq 0$

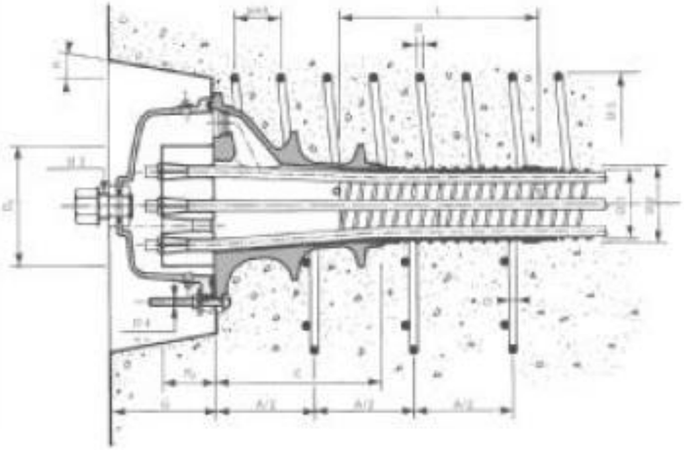
$M_y \neq 0$ $\quad Q_z = 0$

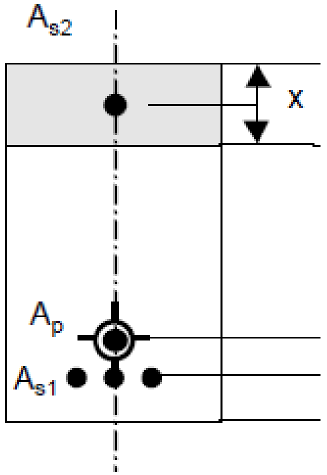
Transverse bending – stress trajectories cont.

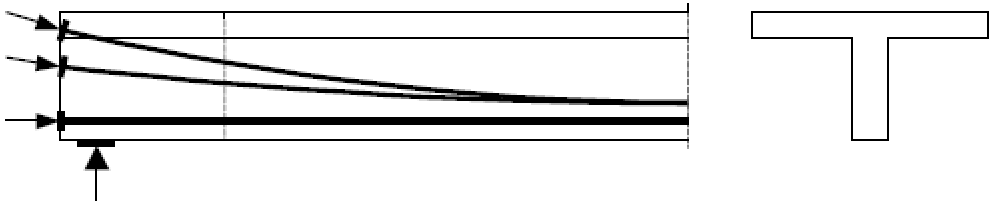




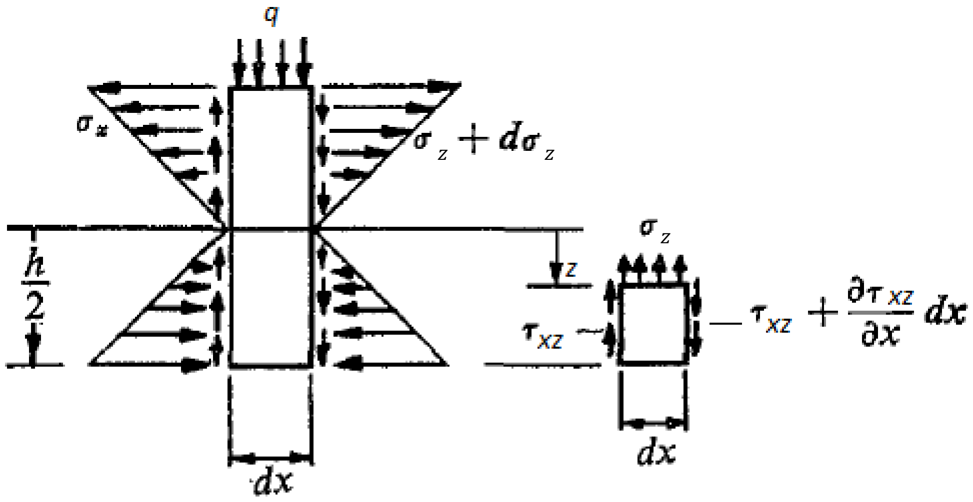








Transverse bending – normal stress σ_z

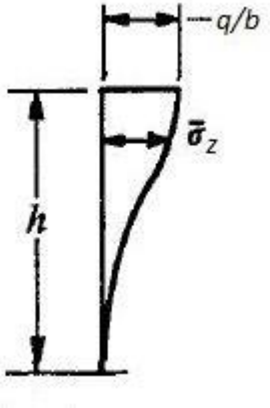


$$\tau_{xz} = \frac{3}{2}\frac{Q}{bh}\left[1-\left(\frac{z}{\frac{h}{2}}\right)^2\right], \qquad \frac{\partial Q}{\partial x} = -q$$

$$b\sigma_z dx = \int_{-\frac{b}{2}}^{\frac{b}{2}}\int_{z}^{\frac{h}{2}} \frac{\partial \tau_{xz}}{\partial x} dxdydz = b\int_{z}^{\frac{h}{2}} \frac{\partial \tau_{xz}}{\partial x} dxdz$$

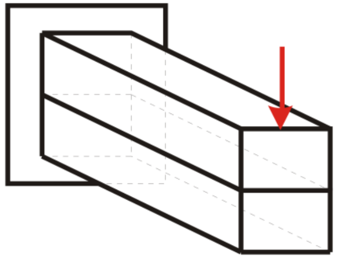
$$\sigma_z = -\int_{z}^{\frac{h}{2}} \frac{3}{2}\cdot\frac{q}{bh}\left[1-\left(\frac{z}{\frac{h}{2}}\right)^2\right]dz = -\frac{q}{b}\left[\frac{1}{2}-\frac{3}{2}\left(\frac{z}{h}\right)+2\left(\frac{z}{h}\right)^3\right]$$

$$\sigma_z\left(\frac{h}{2}\right) = -\frac{q}{b}, \qquad \sigma_z\left(-\frac{h}{2}\right) = 0$$



$$|\sigma_z| \ll |\sigma_x|$$

Transverse bending – shear flow



a)

b)

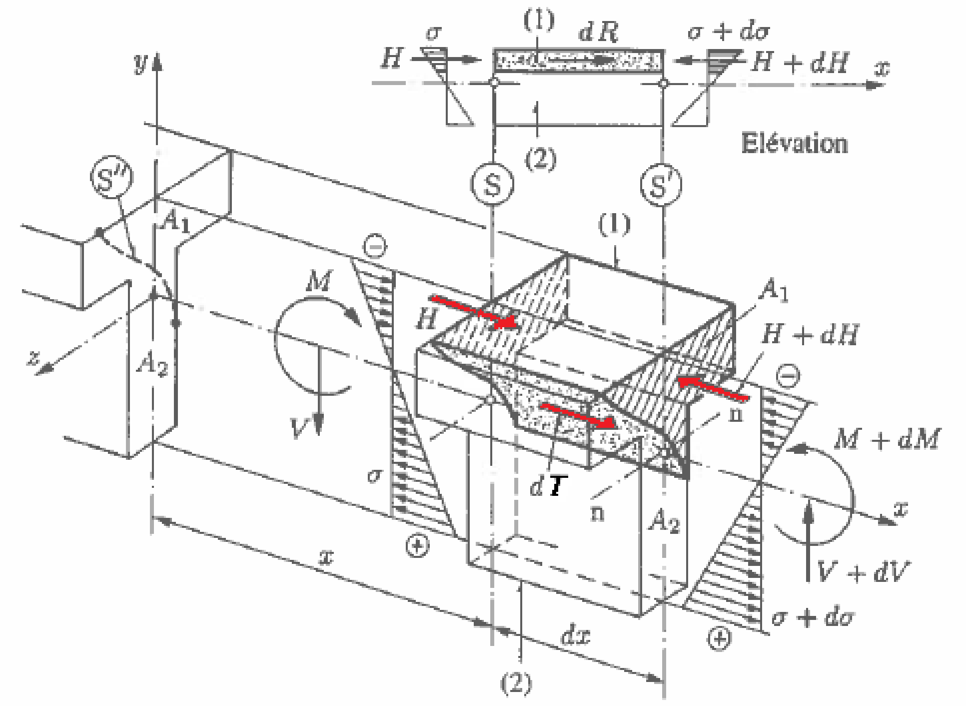
dimensions (total): $b \times h \times l$

$$\text{a)}\quad \text{w} = \text{idem} \rightarrow \kappa = \text{idem} \rightarrow M = \text{idem} = \frac{1}{2}Pl, \quad \sigma_{max} = \frac{M}{W} = \frac{Nl \cdot 12}{2b\left(\frac{h}{2}\right)^3} = 12\frac{Pl}{bh^2}, \quad \kappa = \frac{M}{EI} = \cdots = 48\frac{Pl}{Ebh^3}$$

$$b)\quad M = Pl, \quad \sigma_{max} = \frac{M}{W} = 12\frac{Pl}{bh^3} \cdot \frac{h}{2} = 6\frac{Pl}{bh^2}, \quad \kappa = \frac{M}{EI} = \cdots = 12\frac{Pl}{Ebh^3}$$

(total) shear flow: $T = \tau_{max} bl = \frac{3}{2}\frac{P}{bh} bl = \frac{3}{2}P\frac{l}{h} = \left(\cdots \frac{l}{h} \approx 10 \cdots\right) = 15\ P$

Transverse bending – shear flow (general)



same procedure as previously, but with arbitrary cutting surface

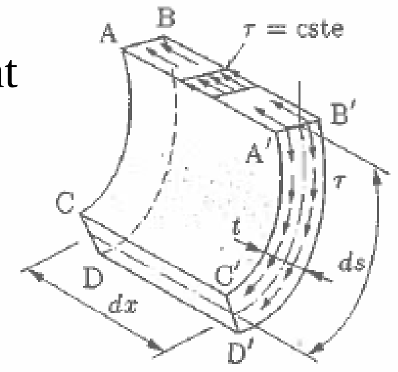
$$dT = \frac{-QS}{I_y}dx, \qquad \tau = \frac{-QS}{I_y t}$$

the shear flow makes the tendency to longitudinal slide between fibers

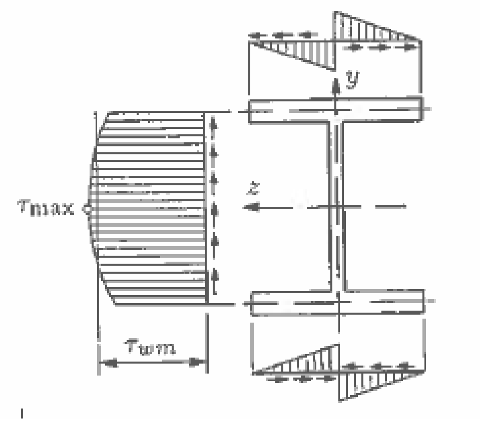
sign „-” means that for growing bending moment *M* the shear force is negative and the shear flow resultant is positive

t – the wall thickness

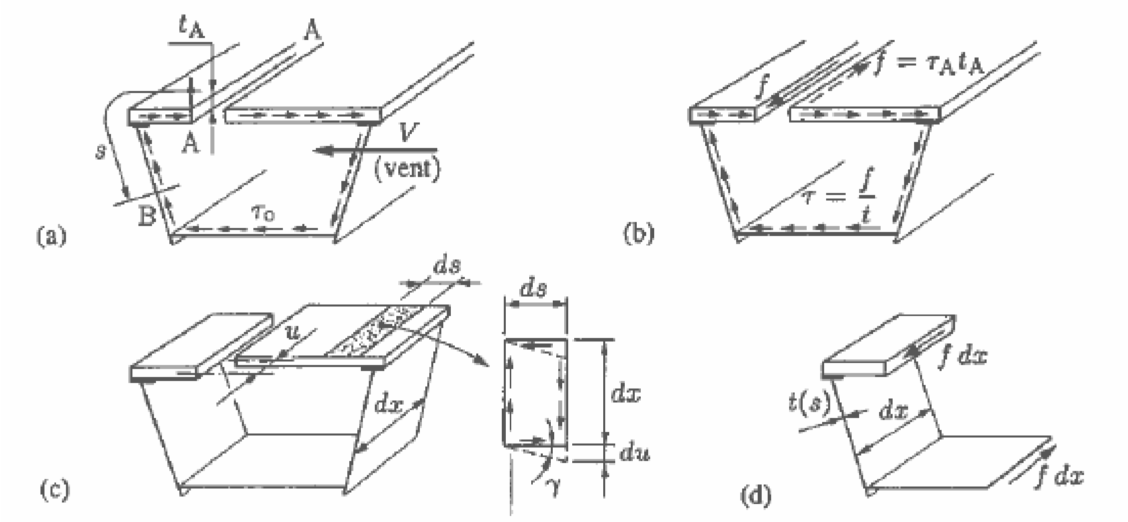
for thin-walled profiles we assume constant value of shear and shear flow stresses along metal sheet thickness and in direction of the central line (a line that bisects the wall thickness)



Shear stress – examples



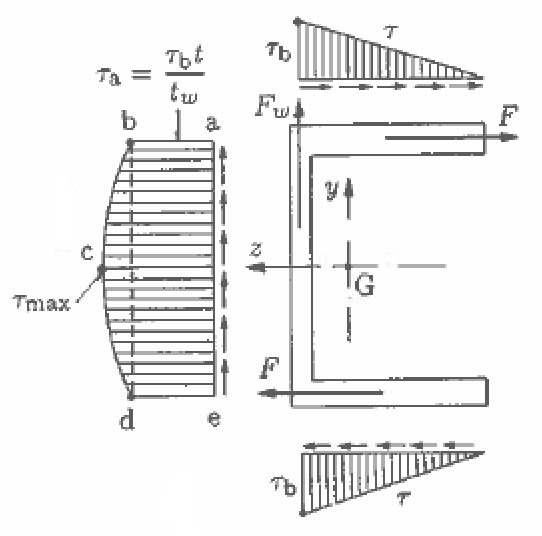
Kirchhoff rule for the stress flows in the „nodes": inflow = outflow
the web transmits ≈ 95 % of the total shear force in the cross-section
shear stress in the web is almost rectangular
steel structures course: $\tau_{wm} = Q/A_{web}$

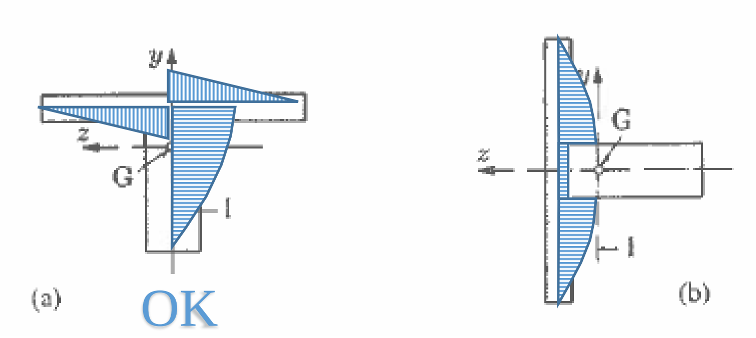


a) we change a closed profile into opened by cutting *AA*, we calculate shear stresses along the central line of profile
c) the stresses cause some relative longitudinal displacement
b) we apply such shear flow *f* that resets the longitudinal displacement; the shear flow causes a constant shear stress along the central line

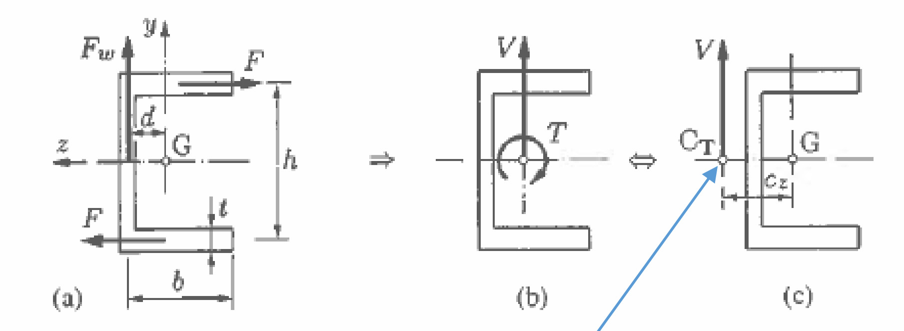
total shear stress = a) + b)

Shear center

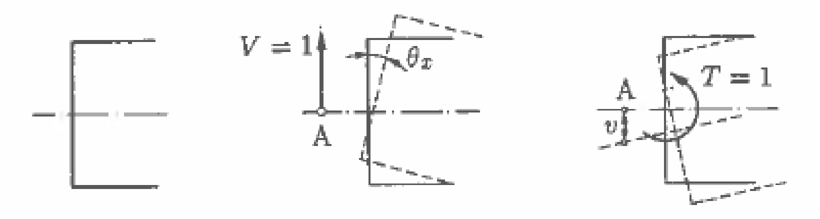




torsional moment ≠ 0



shear center



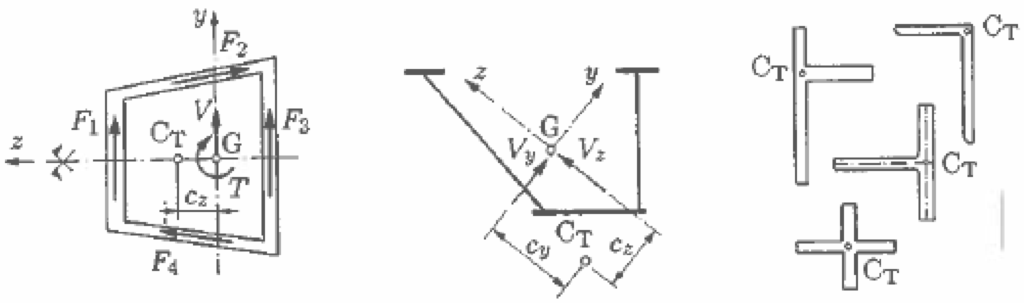
shear center = torsion center

Shear center – cont.

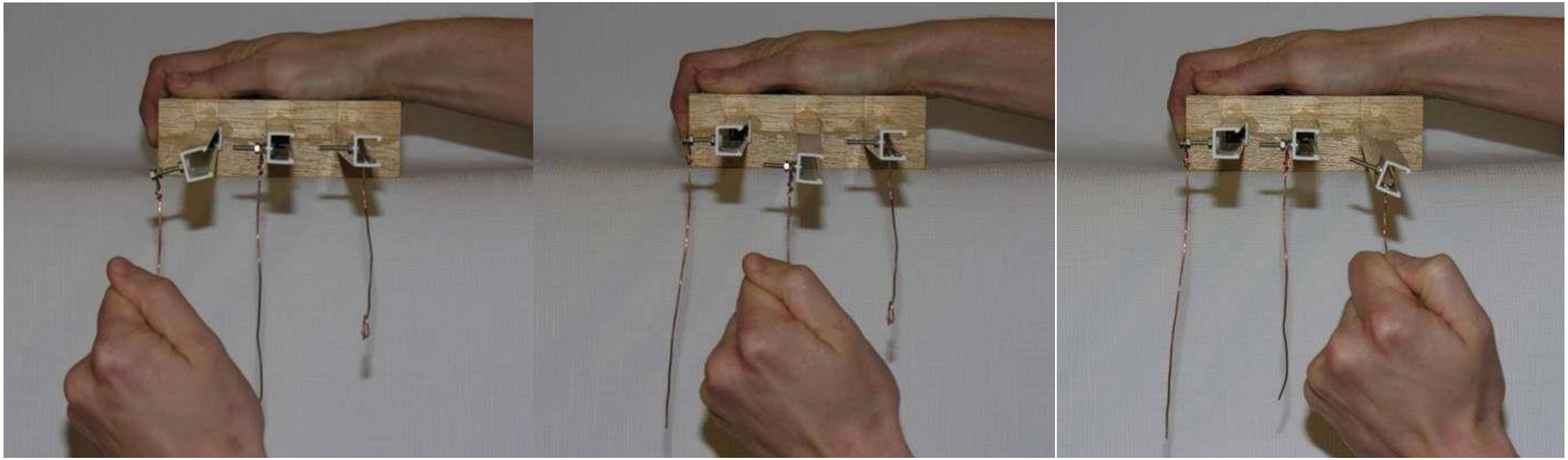
Parallel displacement of the action axis does not change the normal stresses induced by the bending moment. However, the equilibrium condition requires that the action axis of the shear force has a position which coincides with the line of action of the resultant of the shearing stresses. The position of the action axis of the shear force is therefore not arbitrary.

The shear center plays similar role to the transversal forces, as the centroid in relation to the longitudinal (axial) forces. If the resultant axial force passes through the centroid, it will not induce bending. In the same way, if the resultant of the shear forces does not pass through the shear center, it will introduce a torsional moment. The computation of the torsional moment must be made in relation to the shear center, while the bending moment is computed with respect to the centroid.

The shear center should be computed with the use of principal central axes. For the profiles with central lines concurrent at one point however, the position of the shear center is already known and coincides with that point.



Shear center experiment



That’s all for now, folks!