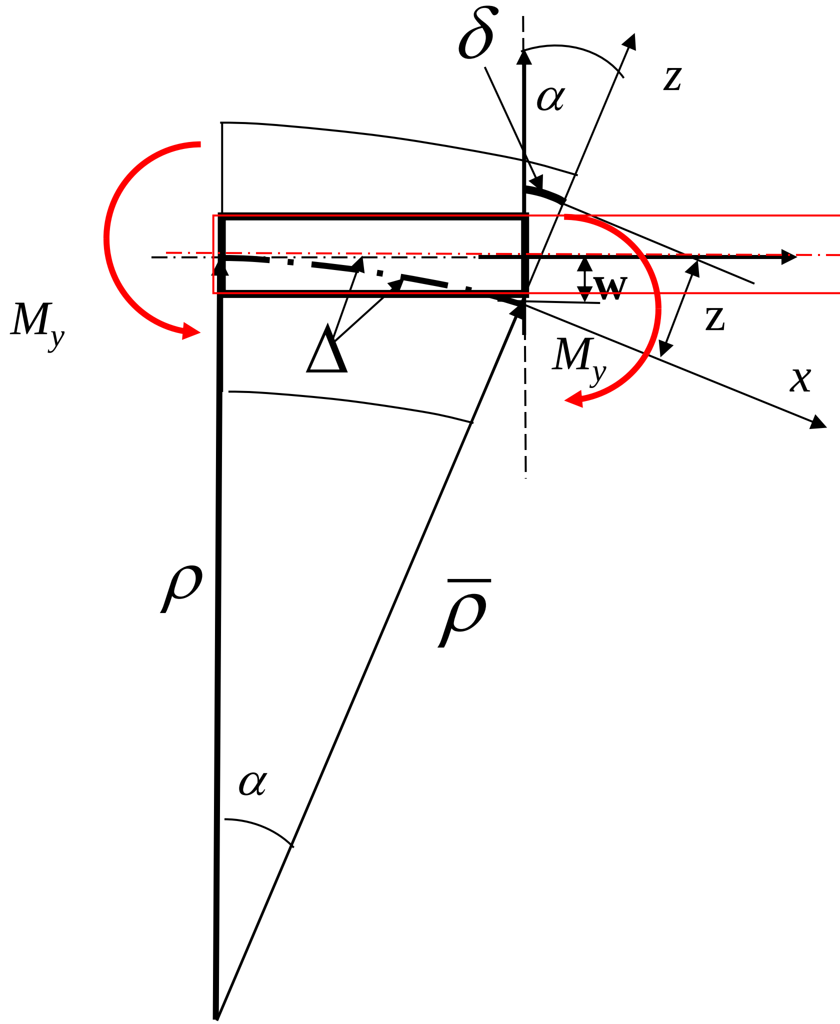
Strength of Materials

6. Beams deflections

Beams deflections – geometry of deformation



1. Strain definition: $\varepsilon_x = \lim_{\Delta \to 0} \frac{\delta}{\Delta}$

2. From the picture: $\delta = \alpha \cdot z$

$\Delta = \alpha \cdot \bar{\rho}$

3. Substitution of 2 into 1 $\varepsilon_x = \lim_{\Delta \to 0} \frac{z}{\bar{\rho}}$

4. Linear strain $\varepsilon_x = \frac{z}{\rho}$

5. Plane cross-section hypothesis $\varepsilon_x = c \cdot z$

$$\boxed{c = \frac{1}{\rho} = \kappa}$$

Beams deflections – normal stress & curv.

1. Normal (linear) strain: $$\varepsilon_x = \frac{z}{\rho}$$

2. Hooke law: $$\sigma_x = E \cdot \varepsilon_x = E\,\tfrac{z}{\rho} = E\kappa z$$

3. Normal stress: $$\sigma_x = \frac{M_y}{J_y} z$$

4. From comparison of 2 and 3: $$\frac{M_y}{J_y} = E\frac{1}{\rho} = E\kappa$$

5. Beam curvature $\boldsymbol{\kappa}$ $$\kappa = \frac{1}{\rho} = \frac{M_y}{EJ_y}$$

Beams deflections – differential equation

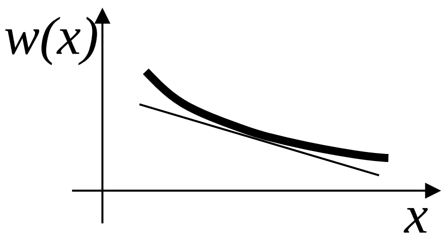
1. Curvature-bending moment relationship: $\kappa = \dfrac{M_y}{EJ_y}$

2. Formula for curvature in mathematics: $$\kappa = \frac{d^2 w / dx^2}{\left[1 + (dw/dx)^2\right]^{3/2}} \cong \frac{d^2 w}{dx^2} \quad (= w'')$$

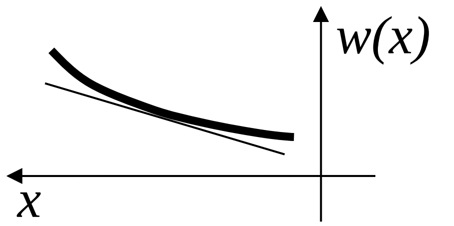
3. Differential equation for beam deflection *w(x)* $$\frac{M_y}{EJ_y} = w''$$

Beams deflections – signs, signs, signs…

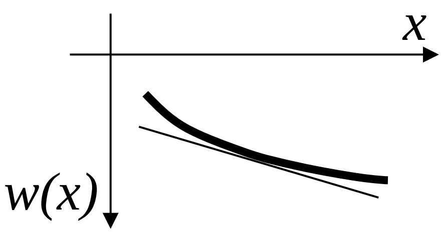
The signs of *w(x), w'(x)* and *w''(x)* depend upon co-ordinate system:



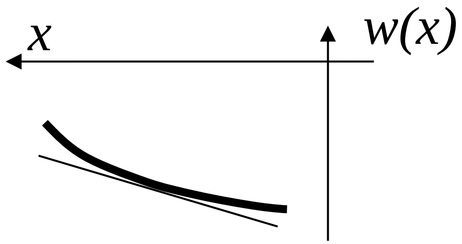
$w>0$, $w'<0$, $w''>0$



$w>0$, $w'>0$, $w''>0$



$w>0$, $w'>0$, $w''>0$

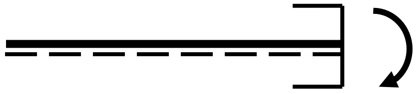


$w<0$, $w'<0$, $w''<0$

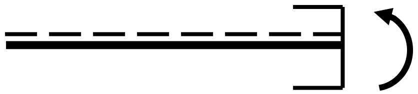
Beams deflections – signs convention

The sign of M(x) follows adopted convention (M is positive when „undersides” are under tension):

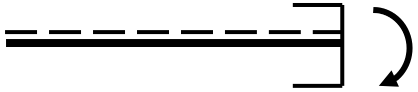
$M>0$



$M<0$



$M<0$



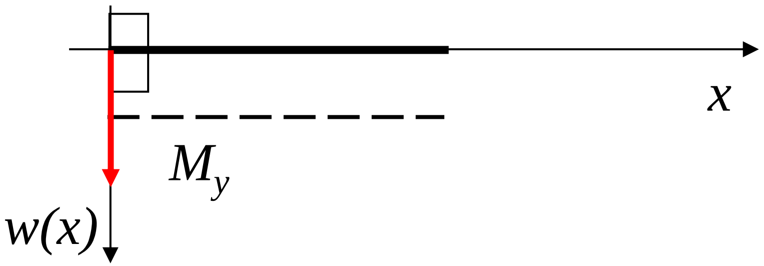
$M>0$

Beams deflections – sign convention cont.

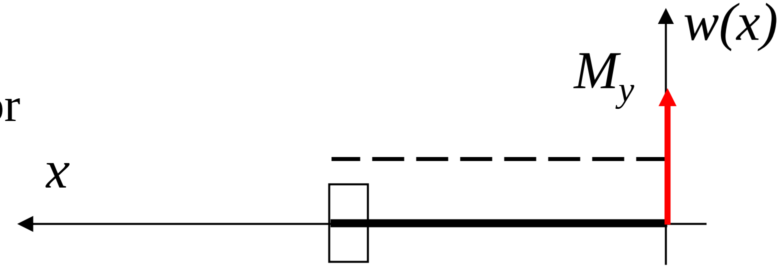
Combination of the previous two conventions will result in following form of the bending deflection equation:

$$\boxed{\frac{M_y(x)}{EJ_y} = -w''(x)}$$

PROVIDED that the positive direction of *w* axis will coincide with positive direction of bending moment axis pointing towards „undersides”:



or



In the opposite case of the two directions discordance the minus sign in the bending equation has to be replaced by the positive sign

The direction of x-axis only does not change the sign in the equation of deflection.
However, one has to remember that in this case the first derivative of deflection *w’* will change the sign!

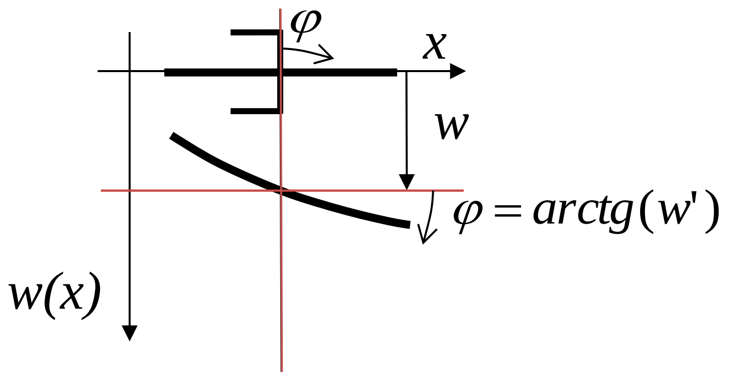
Beams deflections – diff. eq. integration

To find beam deflection one has to integrate the deflection equation twice.

$$w'' = -\frac{M_y}{EJ_y}$$

The first integration yields a tangent to the beam axis and, therefore, rotation of the beam cross-section

$$w' = -\int \frac{M_y}{EJ_y} dx + C$$



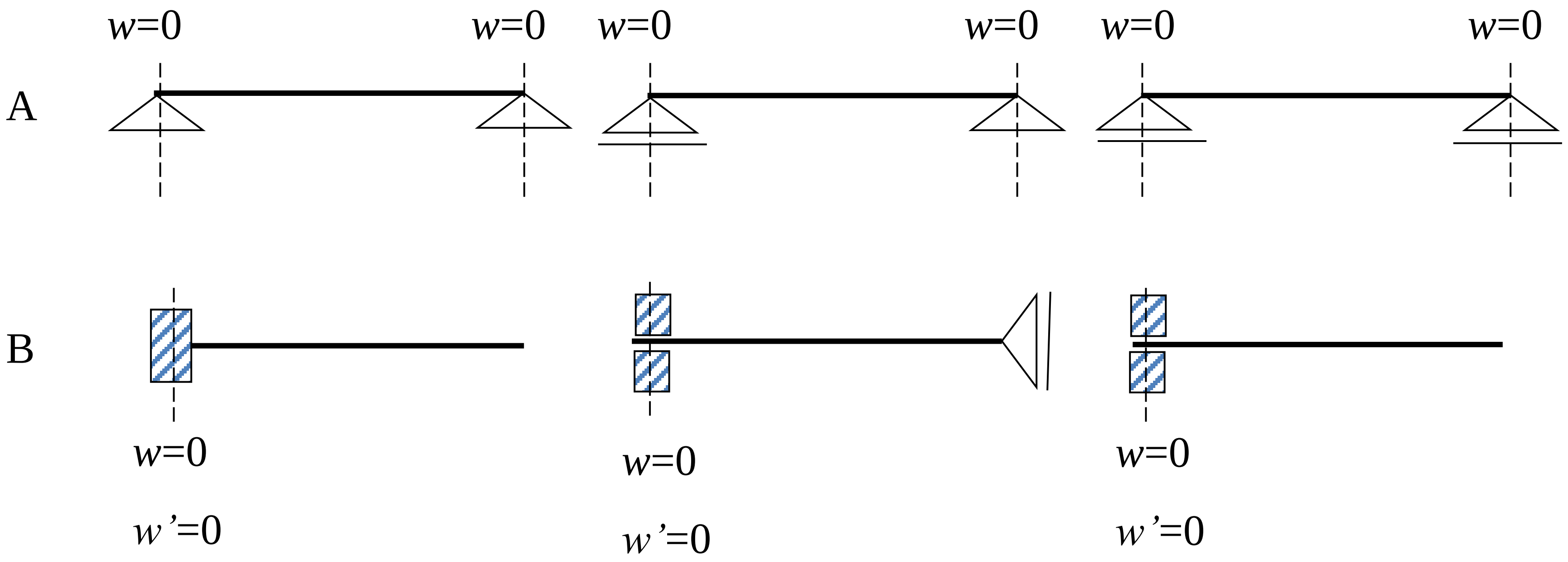
The next integration results in finding beam deflection:

$$w = -\int \left(\int \frac{M_y}{EJ_y} dx \right) dx + Cx + D$$

To determine the values of integration constants C and D we need to formulate boundary conditions

Beams deflections – diff. eq. integration cont.

The boundary conditions have to represent supports of a beam:



NOTICE: As we do not take into account normal forces all three cases shown in the row A are equivalent with respect to deflections calculation. The same is true for the row B.

Beams deflections – boundary conditions

In a general case, when moment bending equation cannot be given by in the analytical form for the whole beam, it has to be formulated and integrated for all characteristic intervals of a beam

As a consequence we need to find 2n constants of integration (where n denotes number of characteristic intervals) and to write down 2n-2 (2 boundary conditions correspond to beam supports) additional compatibility conditions at the neighboring characteristic intervals

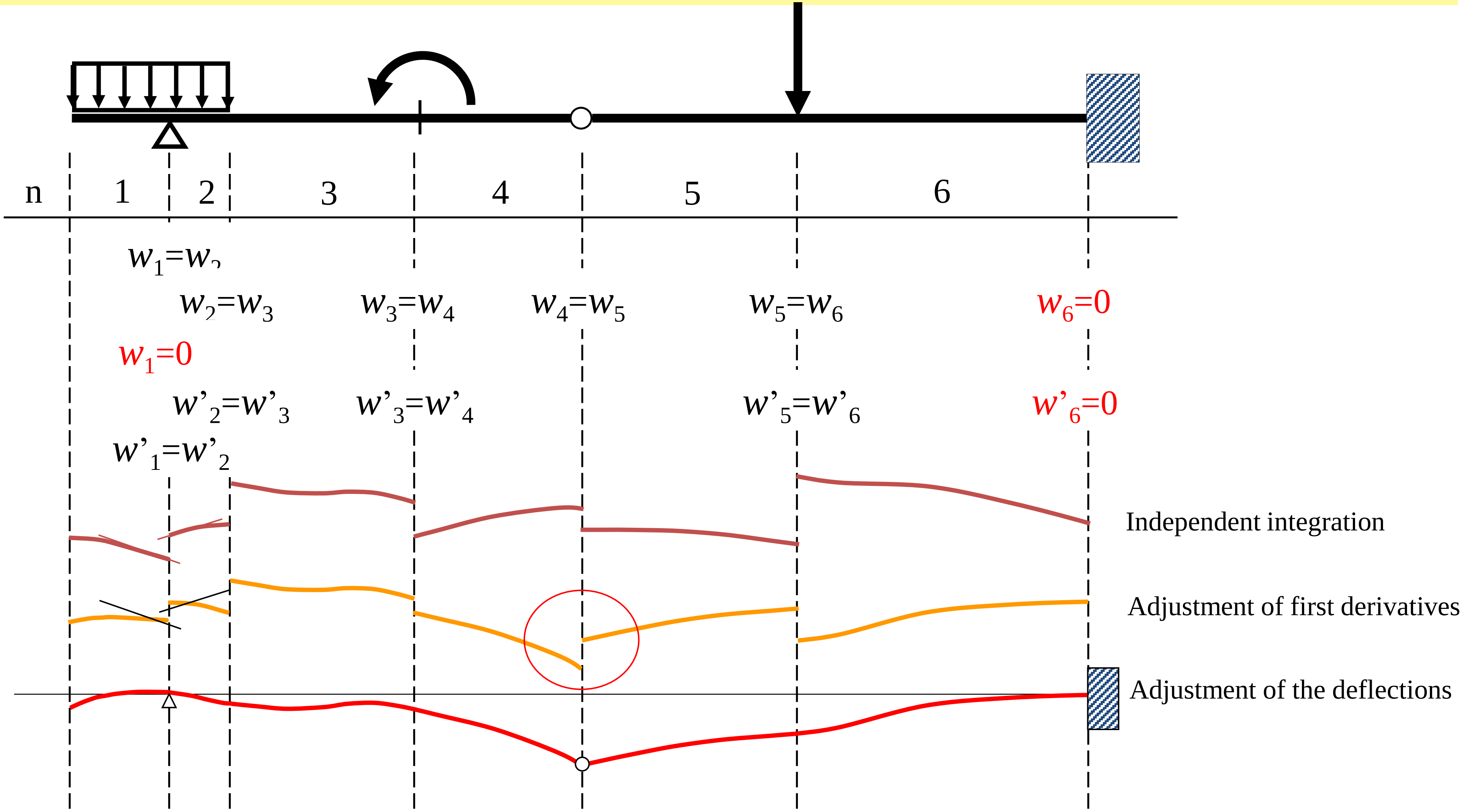
n characteristic intervals $\Rightarrow$ $2n$ integration constants

$2n$ boundary conditions: 2 kinematic boundary conditions + $2(n-1)$ compatibility conditions

The kinematic conditions depend on the supports. The compatibility conditions relate to the deflections and to the rotations compatibility.

This procedure yields the set of 2n linear algebraic equations. The solution of this set can be cumbersome and it is advisable only if an analytical form of a bending deflection for the whole beam is needed.

Beams deflections – compatibility



Beams deflections – Mohr method

STATICS	**DEFLECTIONS**
$$\frac{dM(x)}{dx} = Q_z(x)$$	? $\uparrow$
$$\frac{dQ_z(x)}{dx} = -q(x)$$	? $\uparrow$
$$\frac{d^2M(x)}{dx^2} = -q(x)$$ $\Longleftrightarrow$	$$\frac{d^2w(x)}{dx^2} = -\frac{M(x)}{EJ} \qquad (*)$$
This equation (*) is „integrated" on the basis of ***M(x)* i *Q(x)*** definitions when external loading ***q(x)*** and boundary conditions are known $\Longleftrightarrow$	Why not to use the same method to determine the deflections?

Mohr method; conjugate beam method; fictitious weights method; Mohr fictitious beam method

Mohr method

STATICS - Fictitious domain	**DEFLECTIONS** - Real domain
$$\int\left\{ \frac{d^2 M_F}{dx^2} = -q_F(x) \right\} dx$$	$$\int\left\{ \frac{d^2 w_R}{dx^2} = -\frac{M_R(x)}{EJ} \right\} dx$$
$$\int\left\{ \frac{dM_F}{dx} = Q_F = -\int q_F(x)dx + C_F \right\} dx$$	$$\int\left\{ \frac{dw_R}{dx} = w'_R = -\int \frac{M_R(x)}{EJ}dx + C_R \right\} dx$$
$$M_F(x) = -\int\left[\int q_F(x)dx\right]dx + C_F \cdot x + D_F$$	$$w_R(x) = -\int\left[\int \frac{M_R(x)}{EJ}dx\right]dx + C_R \cdot x + D_R$$

I F

$$\boxed{q_F(x) = \frac{M_R(x)}{EJ} \qquad C_F = C_R \;,\; D_F = D_R}$$

T H E N

$$\boxed{w_R(x) \equiv M_F(x) \;,\; w'_R \equiv Q_F(x)}$$

Mohr method

Fundamental requirement to be satisfied is that fictitious and real beams have the same length ($0 \le x_R \le l$, $0 \le x_F \le l$).

From the condition: $q_F(x) = \frac{M_R(x)}{EI}$ [1/m] follows, that the only loading of the fictitious beams will be continuous loading of the dimension [$Nm/(Nm^{-2}m^4)$] = [m^{-1}] distributed exactly like bending moment distribution for the real beam. Therefore, the bending moment and shear force distributions in the fictitious beams cannot contain any discontinuities (no loading in form of concentrated moments exists and point forces can appear only at the points with articulation).

To satisfy the conditions: $\boldsymbol{C_F = C_R \ , \ D_F = D_R}$

the static conditions have to bet set upon the fictitious beam in such a way that in characteristic points will be:

$$w_R(x) \equiv M_F(x) \ , \ w'_R \equiv Q_F(x)$$

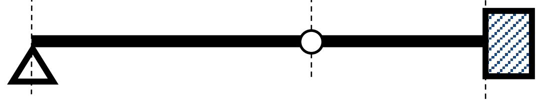
So, if for the real beam $w_R=0$ in a given point, then for the fictitious beam has to be $M_F=0$ in this point. Similarly if $w'_R=0$ then $Q_F=0$ etc.

Summing up, the static boundary conditions for the fictitious beam should correspond to the kinematic boundary conditions of the real beam

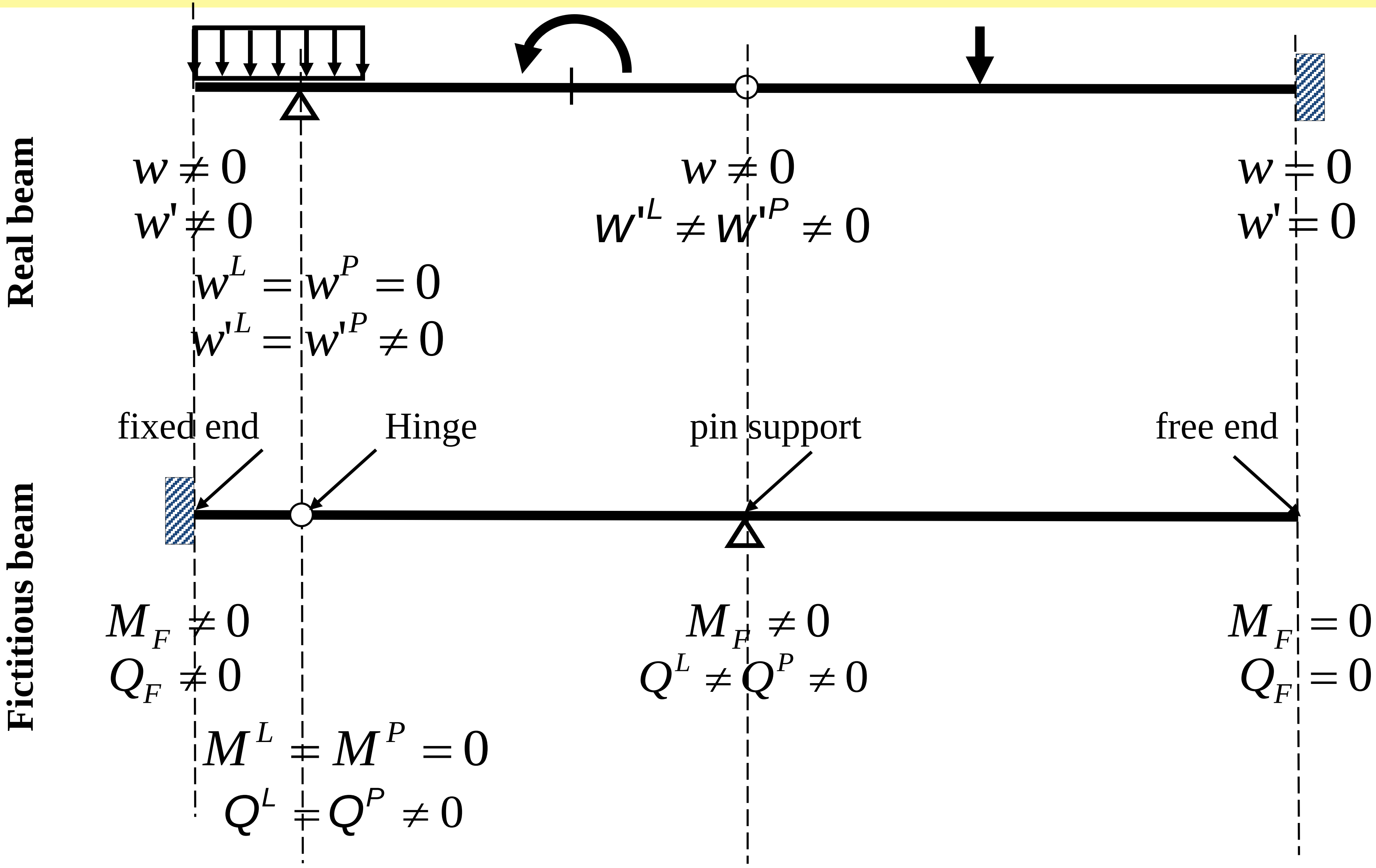
Mohr method cont.

		Example #1		Example #2	
Real beam	Supports $(KBC)_R$				
	Deflection w_R	$=0$	$=0$	$=0$	$\neq 0$
	Rotation w'_R	$\neq 0$	$\neq 0$	$=0$	$\neq 0$
Fictitious beam	Bending moment M_F	$=0$	$=0$	$=0$	$\neq 0$
	Shear force Q_F	$\neq 0$	$\neq 0$	$=0$	$\neq 0$
	Supports $(SBC)_F$				

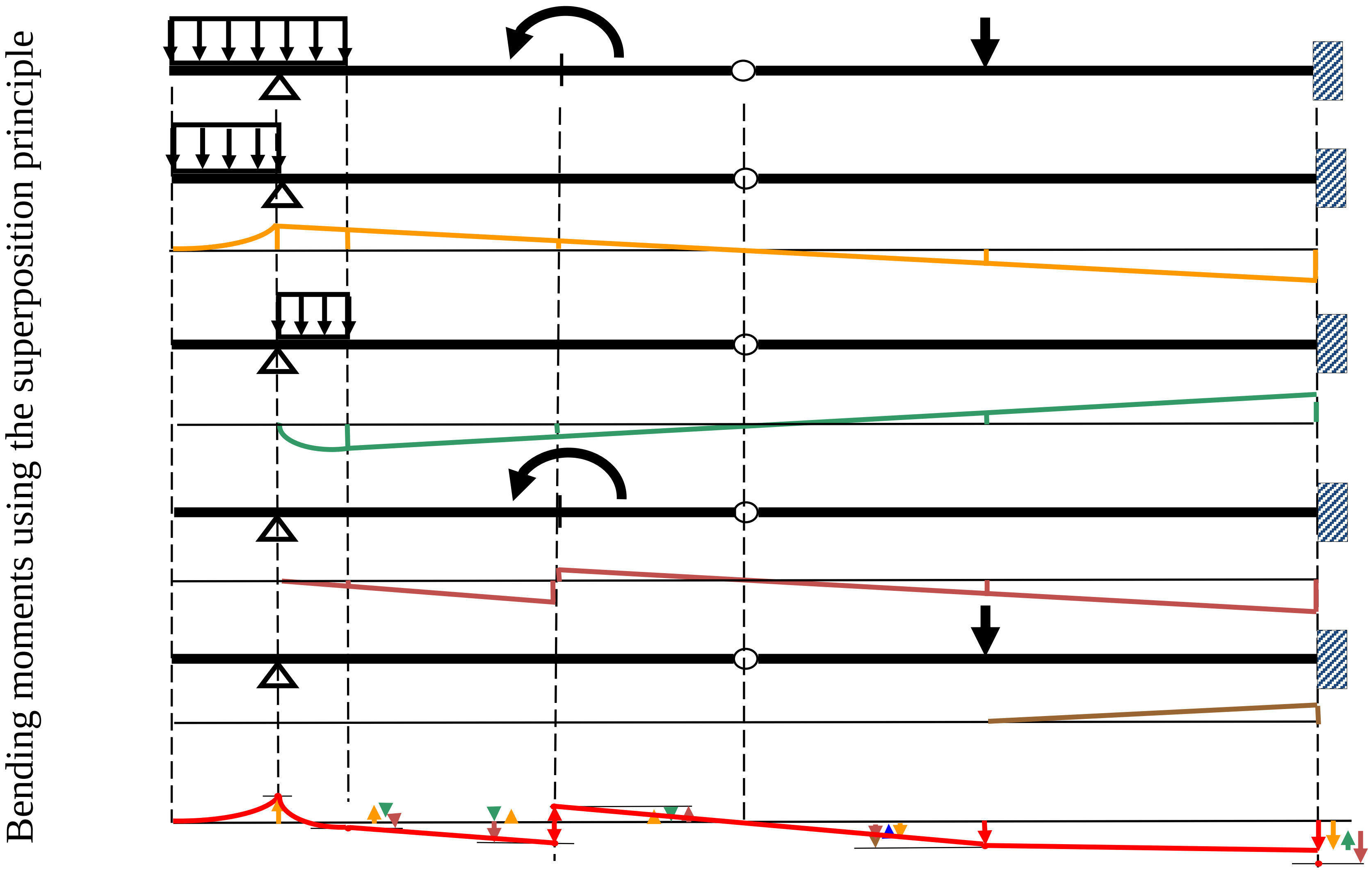
Mohr method cont.

		Example #3	Example #4
Real beam	Supports $(KBC)_R$		
	Deflection w_R	$=0$; $=0$; $\neq 0$	$=0$; $w^L = w^P$, $\neq 0$; $=0$
	Rotation w'_R	$\neq 0$; $w'^L = w'^P$, $\neq 0$; $\neq 0$	$\neq 0$; $w^L \neq w^P$, $\neq 0$; $=0$
Fictitious beam	Bending moment M_F	$=0$; $=0$; $\neq 0$	$=0$; $M^L = M^P$, $\neq 0$; $=0$
	Shear force Q_F	$\neq 0$; $Q^L = Q^P$, $\neq 0$; $\neq 0$	$\neq 0$; $Q^L \neq Q^P$, $\neq 0$; $=0$
	Supports $(SBC)_F$		

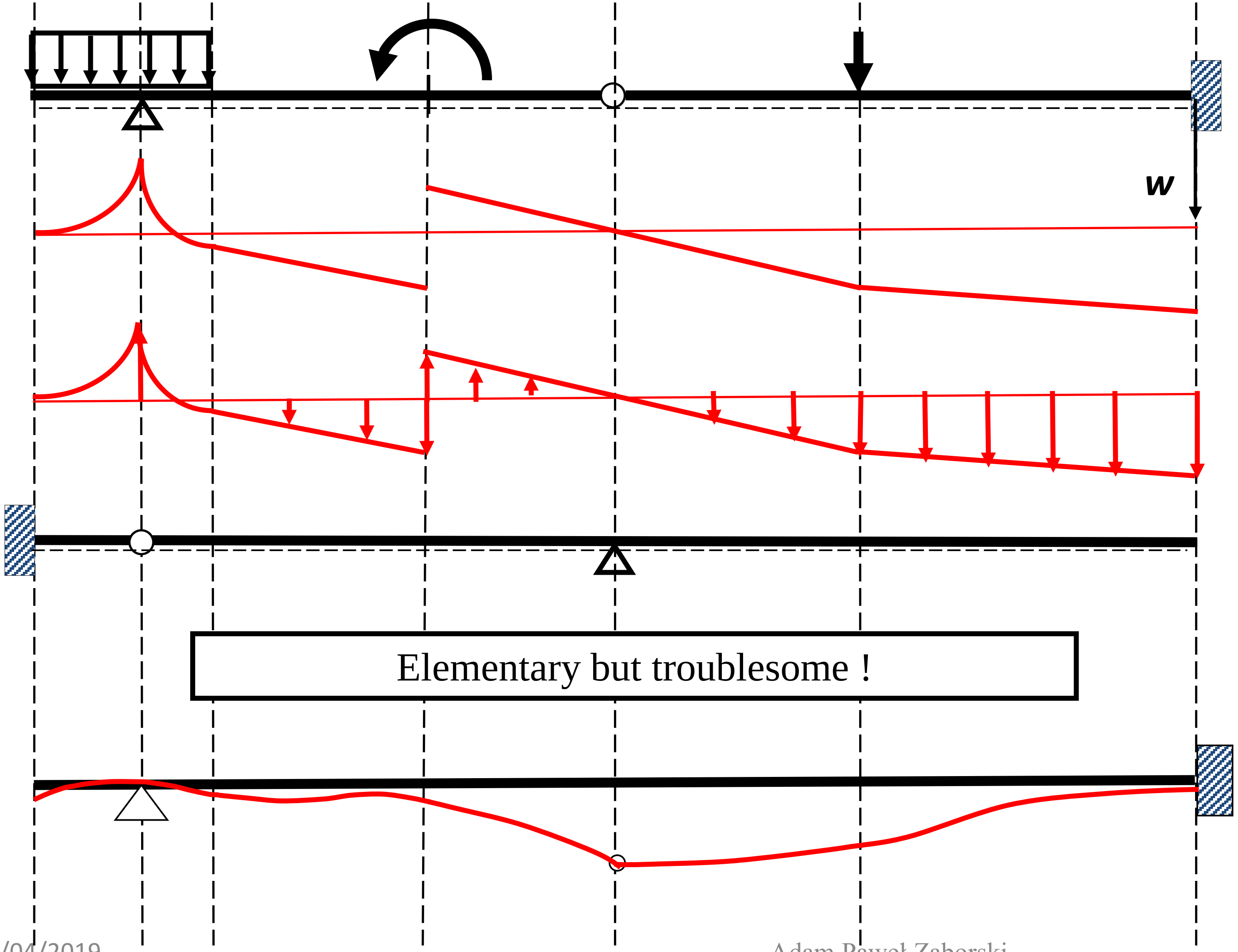
Mohr method – an example



Mohr method – example cont.



Mohr method – cont.



Real beam

Bending moment for the real beam

Fictitious beam loading

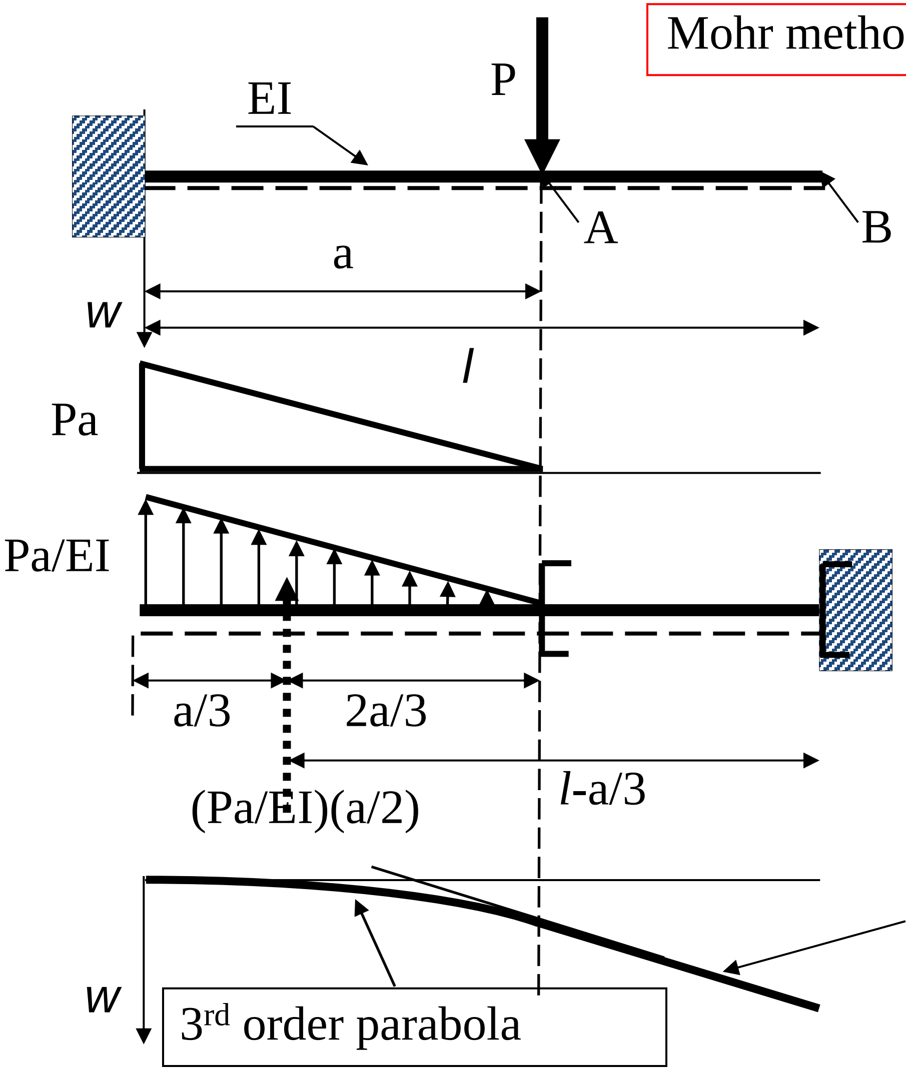
Fictitious beam

Bending moment diagram for fictitious beam

≡ deflections of the real beam

Mohr method – cont.

Mohr method applied to the evaluation of deflections and rotations in chosen points



$w_A = Pa^3/3EI$ $\quad w'_A = Pa^2/2EI$

$w_B = [Pa^2/2EI][l\text{-}a/3]$ $\quad w'_B = Pa^2/2EI$

Bending moments for the fictitious beam

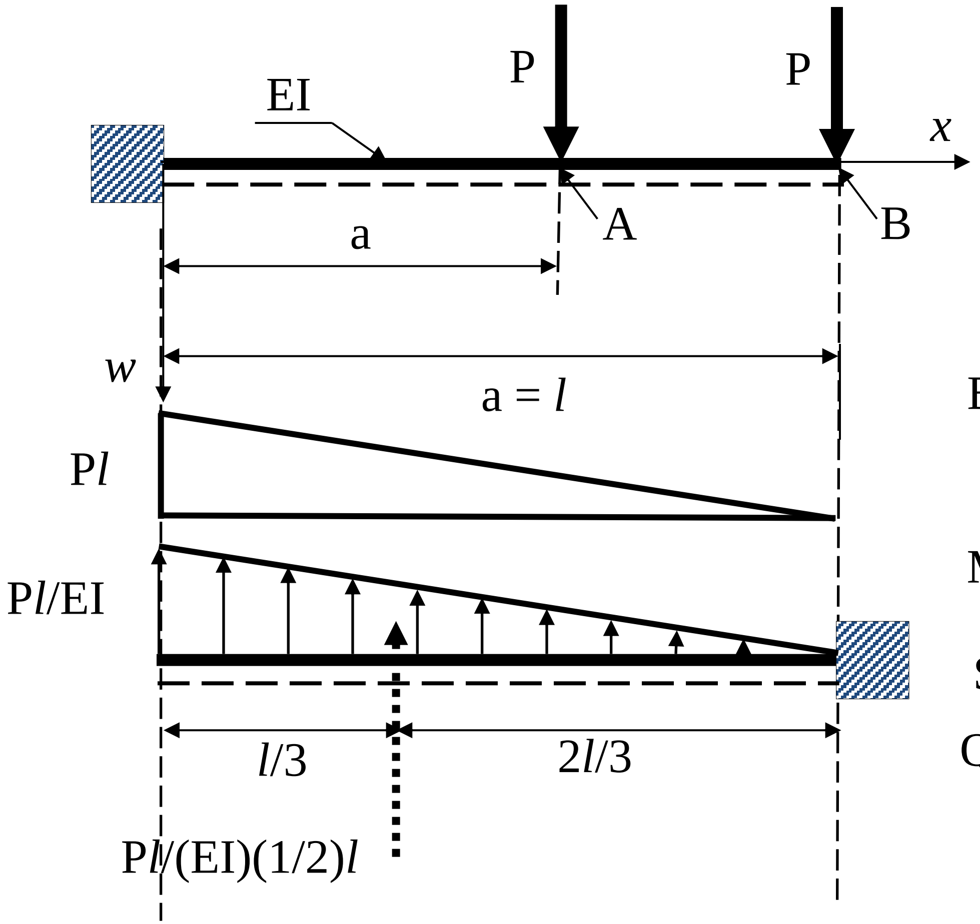
$M^A{}_F = w_A = (Pa/EI)(a/2)(2a/3) = \quad Pa^3/3EI$

$M^B{}_F = w_B = (Pa/EI)(a/2)[l\text{-}a/3] = \quad [Pa^2/2EI][l\text{-}a/3]$

Shear forces for the fictitious beam:

$Q^A{}_F = Q^B{}_F = w'_A = w'_B = (Pa/EI)(1/2Ia = \ Pa^2/2EI$

Mohr method – example



Special case: a = l

$\boxed{w_B = Pl^3/3EI = w_{max}}$

$w'_B = Pl^2/2EI$

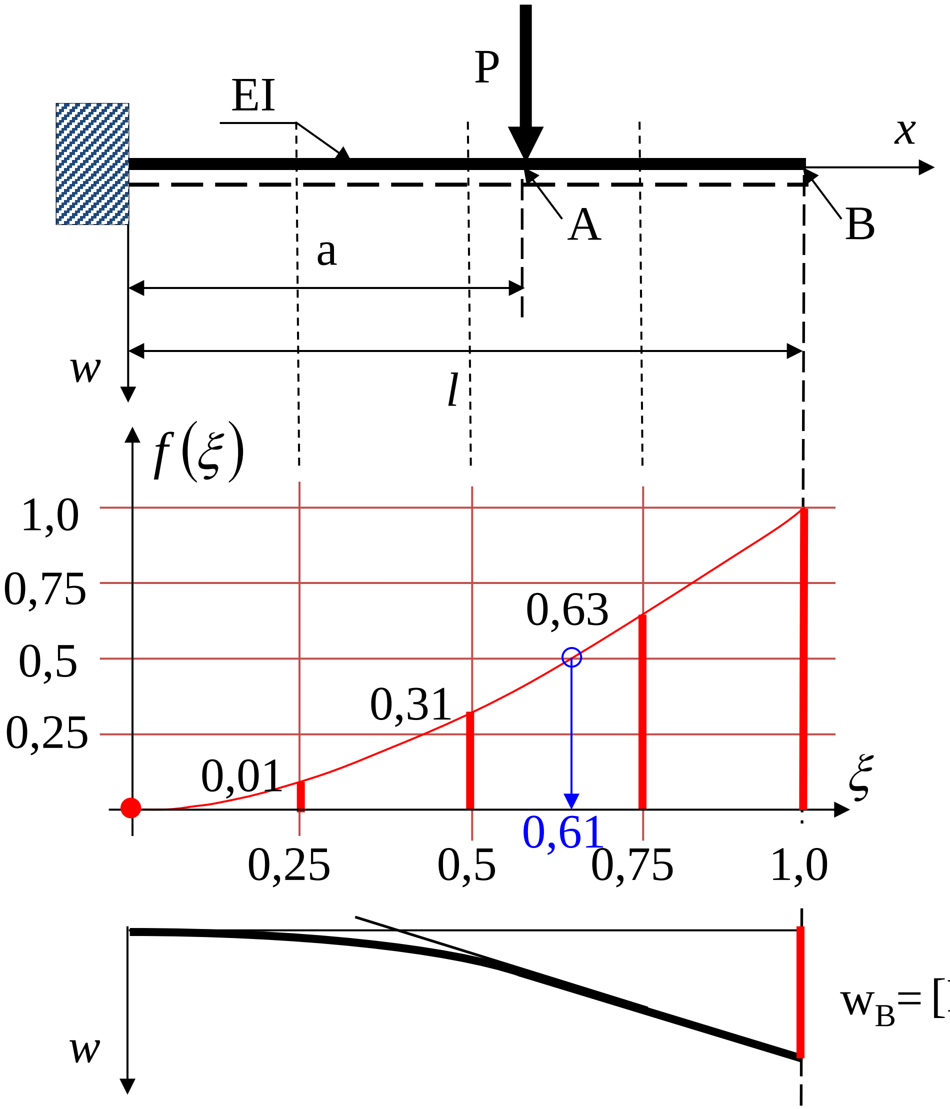
Bending moment for the fictitious beam

$M^B_F = w_B = Pl/(EJ)(1/2)l[2l/3] = Pl^3/3EI$

Shear forces for the fictitious beam

$Q^B_F = w'_B = Pl/(EI)(1/2)l = Pl^2/2EI$

Mohr method – cont.



w_B=[Pa^2/2EJ][l-a/3]

Special example: end beam deflection as the function of force position

$$w_B = \frac{Pa^2}{3EI}\left(1 - \frac{a}{3l}\right) l \frac{3l^2}{2l^2}$$

$$w_B = \frac{Pl^3}{3EI}\left(1 - \frac{a}{3l}\right) \frac{3a^2}{2l^2}$$

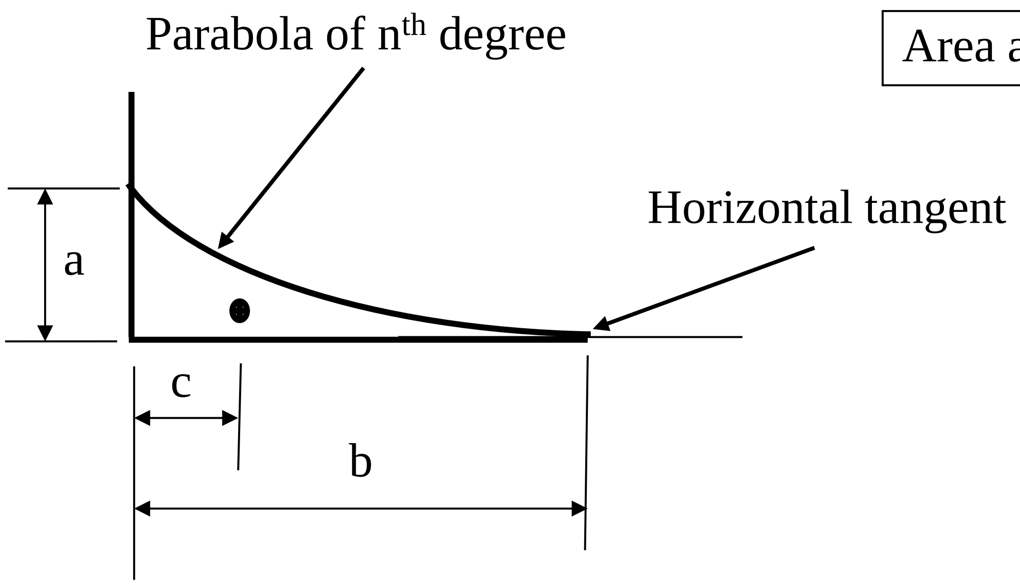
$$\boxed{\xi = \frac{a}{l}}$$

w_{max}

$$w_B = \frac{Pl^3}{3EI}\left(1 - \frac{\xi}{3}\right)\frac{3}{2}\xi^2$$

$$\boxed{w_B = f(\xi) \cdot w_{\max}}$$

Mohr method – some formulae



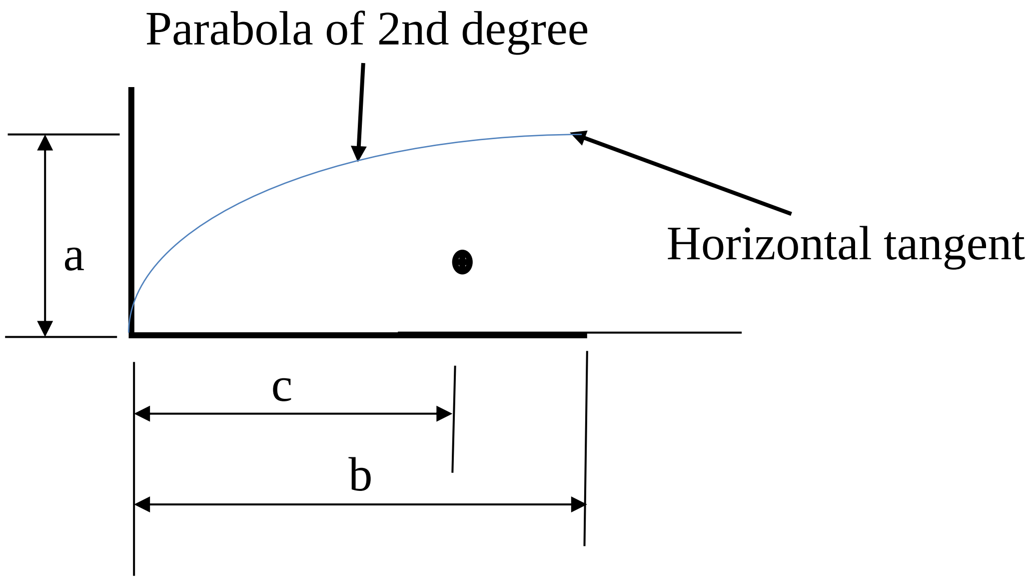
Area and center of gravity for parabolic shapes

Area: $A = \frac{ab}{n+1}$

Position of the center of gravity: $c = \frac{b}{n+2}$

n	**A**	**c**
0	ab	$b/2$
1	$ab/2$	$b/3$
2	$ab/3$	$b/4$
3	$ab/4$	$b/5$
…	…	…

Mohr method – some formulae



Area: $A = \frac{2}{3}ab$

Position of the center of gravity: $c = \frac{5}{8}b$

Beams deflections – general bending eq.

Let us differentiate twice the equation: $\frac{d^2w}{dx^2} = -\frac{M_y}{EI_y}$ making use of relationships $\frac{dM_y}{dx} = Q_z$ $\frac{dQ_z}{dx} = -q$

The first **differentiation** yields:

$$\frac{d}{dx}w'' = -\frac{d}{dx}\frac{M_y}{EI_y} \longrightarrow \frac{d^3w}{dx^3} = -\frac{dM_y}{dx}\frac{1}{EI_y} \longrightarrow \boxed{\frac{d^3w}{dx^3} = -\frac{Q_z}{EI_y}}$$

The second **differentiation**:

$$\frac{d}{dx}\frac{d^3w}{dx^3} = -\frac{d}{dx}Q_z\frac{1}{EI_y} \longrightarrow \boxed{\frac{d^4w}{dx^4} = \frac{q}{EI_y}}$$

Double integration of the equation *w''*= - *M/EI* allows for finding rotations and deflections:

$$\boxed{w' = -\int\frac{M_y}{EJ_y}dx + C \qquad w = \int\left(\int\frac{M_y}{EJ_y}dx\right)dx + Cx + D}$$

Beams deflections – overall picture of the problem

Loading $$\frac{d^4w}{dx^4} = q\frac{1}{EI_y}$$

Shear force $$\frac{d^3w}{dx^3} = -Q_z\frac{1}{EI_y}$$

DIFFERENTIATION

INTEGRATION

Bending moment $$\boxed{\frac{d^2w}{dx^2} = -\frac{M_y}{EI_y}}$$

$$w' = -\int \frac{M_y}{EI_y}dx + C \qquad w = -\int\left(\int \frac{M_y}{EI_y}dx\right)dx + Cx + D$$

Curvature

Rotation

Deflection

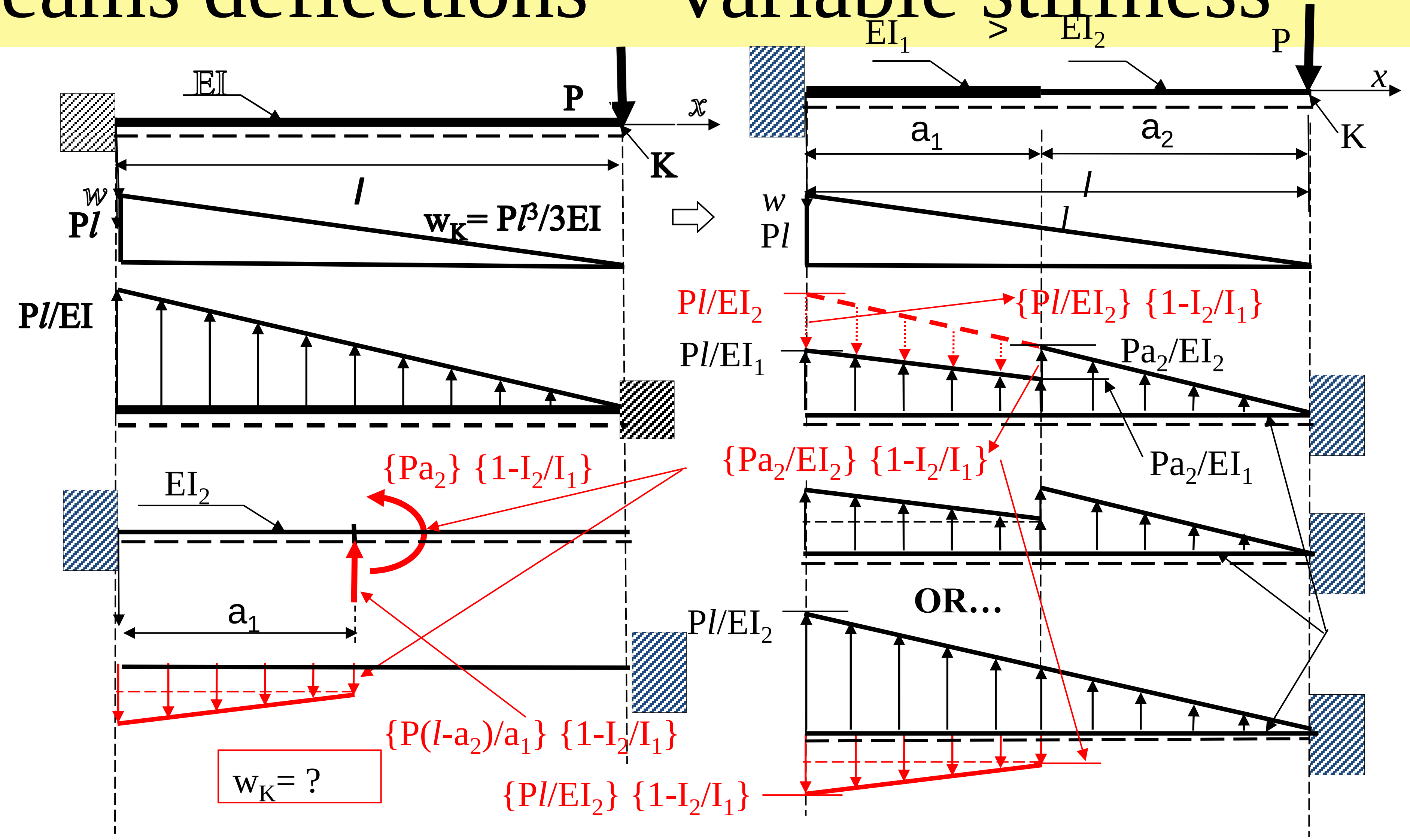
Beams deflections – variable stiffness

If EI changes along beam axis, $EI(x)$, then differential equation for displacement becomes:

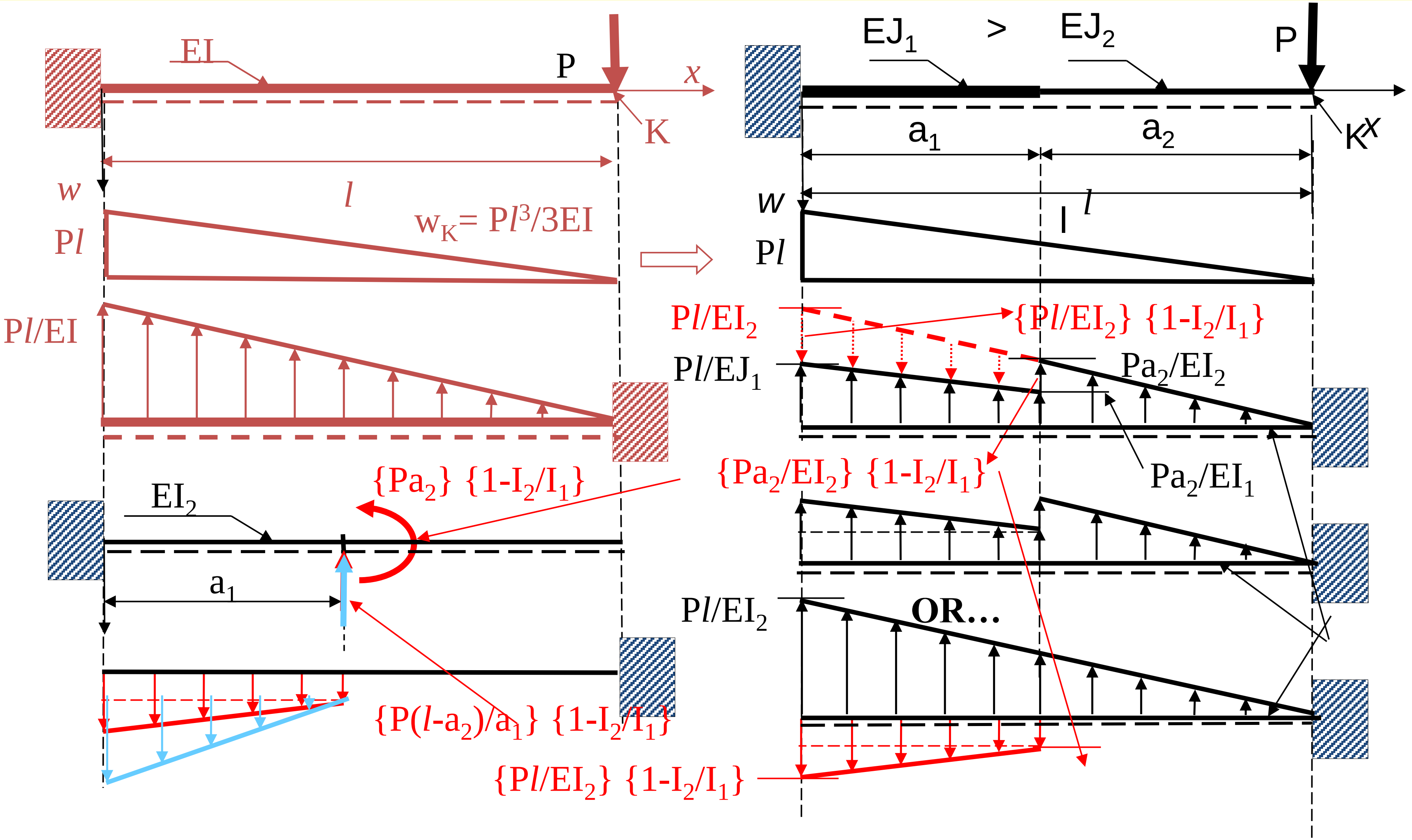
$$\frac{d^2 w(x)}{dx^2} = -\frac{M(x)}{EI(x)} \quad \Longrightarrow$$

Therefore the points in which bending stiffness sharply changes are also characteristic points.

Beams deflections – variable stiffness



Beams deflections – variable stiffness



Beams deflections – Macaulay’s method

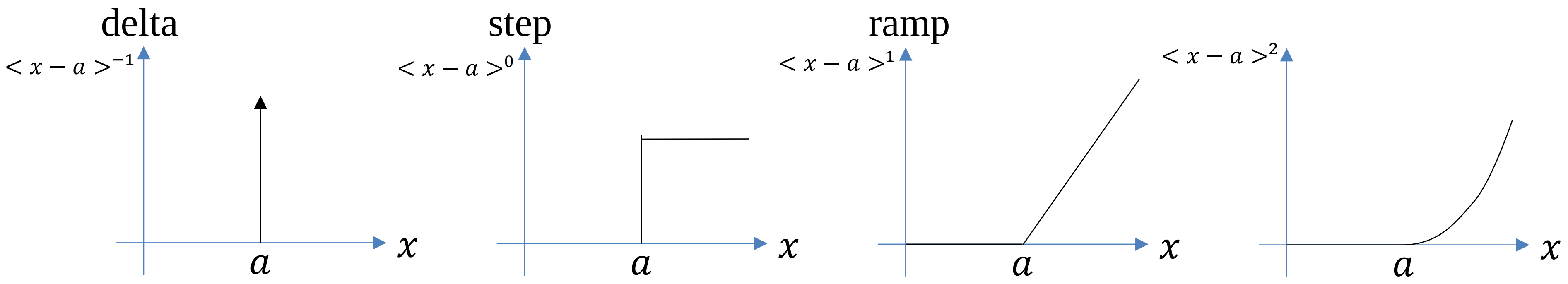
The method is based on the concept of discontinuity functions (sometimes called as half-range functions or generalized functions or distribution functions).

The Dirac delta function: $\delta(x-a) =< x-a >^{-1}= \begin{cases} 0, & x \neq a \\ \infty, & x \to a \end{cases}$, $\int_{-\infty}^{x} < x-a >^{-1} dx = 1$

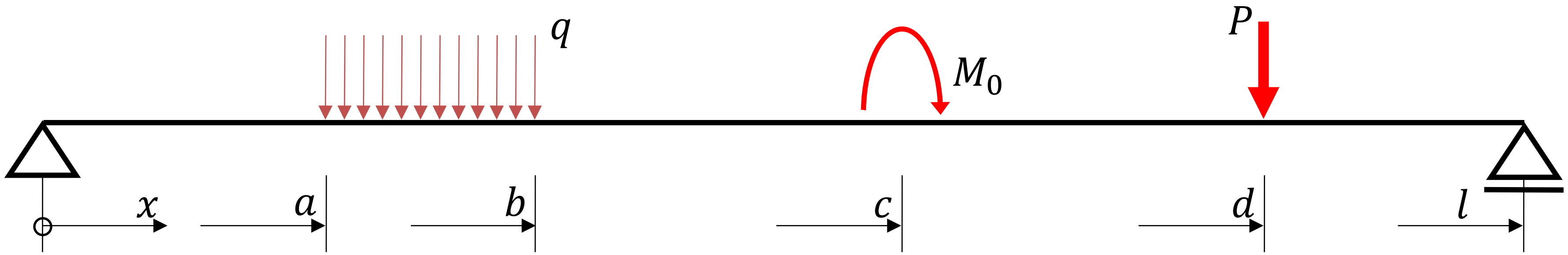
The Heaviside step function: $H(x-a) = \langle x-a \rangle^0 = \int_{-\infty}^{x} < x-a >^{-1} dx = \begin{cases} 0, & x < a \\ 1, & x > a \end{cases}$

The ramp function: $< \mathrm{x} - \mathrm{a} >^1 = \int_{-\infty}^{x} < x-a >^0 dx = \begin{cases} 0, & x < a \\ x-a, & x > a \end{cases}$

and similarly: $< x-a >^n = \begin{cases} 0, & x \leq a \\ (x-a)^n & x > a \end{cases}$, $\boxed{\int_{-\infty}^{x} < x-a >^n dx = \frac{(x-a>^{n+1}}{n+1}, \quad n \geq 0}$



Beams deflections – Macaulay's method example



$$M(x) = R_A < x - 0 >^1 - \frac{q < x - a >^2}{2} + \frac{q < x - b >^2}{2} + M_0 < x - c >^0 - P < x - d >^1$$

$$EIw''(x) = -M(x) = -R_A x + \frac{q < x - a >^2}{2} - \frac{q < x - b >^2}{2} - M_0 < x - c >^0 + P < x - d >$$

$$EIw'(x) = C - \frac{1}{2} R_A x^2 + \frac{q < x - a >^3}{6} - \frac{q < x - b >^3}{6} - M_0 < x - c > + \frac{1}{2} P < x - d >^2$$

$$EIw(x) = Cx + D - \frac{1}{6} R_A x^3 + \frac{q < x - a >^4}{24} - \frac{q < x - b >^4}{24} - \frac{1}{2} M_0 < x - c >^2 + \frac{1}{6} P < x - d >^3$$

KBC: $w(0) = 0 \rightarrow 0 = D - 0 + 0 - 0 - 0 + 0 = D \rightarrow D = 0$

$$w(l) = 0 \rightarrow 0 = Cl - \frac{1}{6} R_A l^3 + \frac{q(l - a)^4}{24} - \frac{q(l - b)^4}{24} - \frac{1}{2} M_0 (l - c)^2 + \frac{1}{6} P(l - d)^3 \rightarrow C = \cdots$$

$$w(c < x_1 < d) = \frac{1}{EI}\left[Cx_1 - \frac{1}{6} R_A x_1^3 + \frac{q(x_1 - a)^4}{24} - \frac{q(x_1 - b)^4}{24} - \frac{1}{2} M_0 (x_1 - c)^2 + 0\right] = \cdots$$

Macaulay's method – cont.

$$M(x) = R_A < x - 0 >^1 - \frac{q < x - a >^2}{2} + \frac{q < x - b >^2}{2} + M_0 < x - c >^0 - P < x - d >^1$$

alternative notation: Clebsch's method

$$M(x) = R_A x\Big|_a - \frac{q(x-a)^2}{2}\Bigg|_b + \frac{q(x-b)^2}{2}\Bigg|_c + M_0(x-c)^0\Big|_d - P(x-d)\Big|_l$$

student's slang: a barrier

If you see a bar set down, you should stop! If the current coordinate is less than the value written at the bar, you stop.

If the value is greater than the bar value, you can cross the bar.

what it means really? Five equations in one!

Macaulay's method – cont.

$$M(x) = R_A x\Big|_a - \frac{q(x-a)^2}{2}\Big|_b + \frac{q(x-b)^2}{2}\Big|_c + M_0(x-c)^0\Big|_d - P(x-d)\Big|_l$$

it means:

$M(x) = R_A x|_a$ $\rightarrow$ for the first characteristic interval, $x \leq a$

$M(x) = R_A x|_a - \frac{q(x-a)^2}{2}\Big|_b$ $\rightarrow$ for the second characteristic interval, $x \leq b$

$M(x) = R_A x|_a - \frac{q(x-a)^2}{2}\Big|_b + \frac{q(x-b)^2}{2}\Big|_c$ $\rightarrow$ for the third characteristic interval, $x \leq c$

$M(x) = R_A x|_a - \frac{q(x-a)^2}{2}\Big|_b + \frac{q(x-b)^2}{2}\Big|_c + M_0(x-c)^0|_d$ $\rightarrow$ for the fourth characteristic interval, $x \leq d$

$M(x) = R_A x|_a - \frac{q(x-a)^2}{2}\Big|_b + \frac{q(x-b)^2}{2}\Big|_c + M_0(x-c)^0|_d - P(x-d)|_l$ $\rightarrow$ for the last characteristic interval

the method can be viewed as a smart way to write down several equations in the form of general one equation to avoid possible errors, the integration constants are written first, from the beginning of equation (because they are common for all equations), e.g.:

$$EIw(x) = Cx + D - \frac{1}{6}R_A x^3\Big|_a - \frac{q(x-a)^4}{24}\Big|_b + \frac{q(x-b)^4}{24}\Big|_c + \frac{1}{2}M_0(x-c)^2\Big|_d - \frac{1}{6}P(x-d)^3\Big|_l$$

Beams deflections – virtual works principle

The principle of virtual works (virtual displacements, real forces)

balance of a set of real forces or stresses

$$\int_{S_t} \bar{\boldsymbol{u}}^{\boldsymbol{T}} \boldsymbol{T}\, \mathrm{dS} + \int_V \bar{\boldsymbol{u}}^{\boldsymbol{T}} \boldsymbol{f} \mathrm{dV} = \int_V \bar{\boldsymbol{\varepsilon}}^{\boldsymbol{T}} \boldsymbol{\sigma} \mathrm{dV}$$

S_t – boundary with the static conditions, $\boldsymbol{T}$ – external forces, $\boldsymbol{f}$ – mass forces, $\boldsymbol{\varepsilon}$, $\boldsymbol{\sigma}$ – strain and stress tensors, the bar means virtual quantity

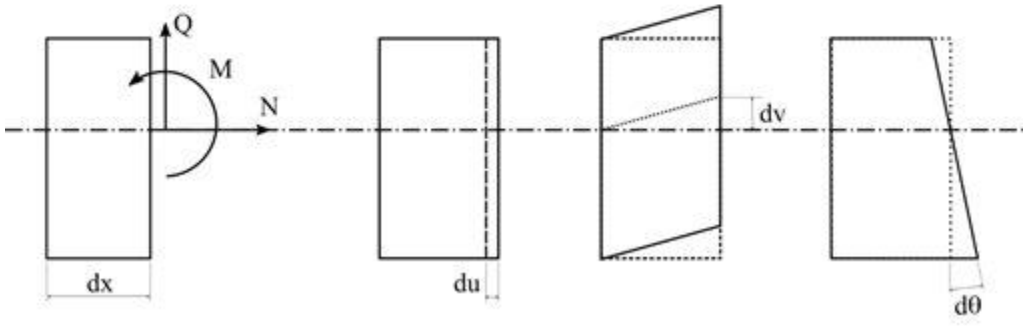
The principle of complementary virtual works (virtual forces, real displacements)

$$\int_{S_t} \boldsymbol{u}^{\boldsymbol{T}} \bar{\boldsymbol{T}}\, \mathrm{dS} + \int_V \boldsymbol{u}^{\boldsymbol{T}} \bar{\boldsymbol{f}} \mathrm{dV} = \int_V \boldsymbol{\varepsilon}^{\boldsymbol{T}} \bar{\boldsymbol{\sigma}} \mathrm{dV}$$

valid for any constitutive equation and finite displacements

Virtual works principle – cont.

prismatic bars:
correspondence between each cross-sectional force and one degree of freedom:



elongation
transverse displacement
rotation angle

Cross-sectional force	stress	strain	displacement
N	$\sigma = \frac{N}{A}$	$\varepsilon = \frac{du}{dx}$	$du = \varepsilon dx$
Q	$\tau = \frac{QS}{Ib}$	$\beta = \frac{dv}{dx}$	$dv = \beta dx$
M_y	$\sigma = \frac{M}{I_y} z$	$\kappa = \frac{d^2 w}{dx^2}$	$d^2 w = \kappa dx^2$
M_x	$\tau = \frac{M_x}{I_x} t, \frac{M_x}{2\Omega\delta}, \dots$	$\varphi = \frac{d\theta}{dx}$	$d\theta = \alpha dx$

virtual work:

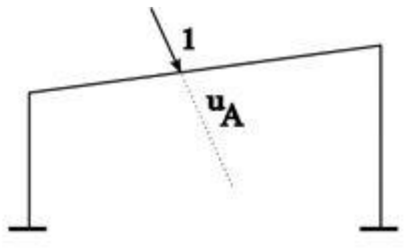
$$\overline{W}_w = \int_L \left(\overline{N}\varepsilon + \overline{Q}\beta + \overline{M_y}\kappa \right) dx$$

or

$$\overline{W}_w = \int_L \left(N\bar{\varepsilon} + Q\bar{\beta} + M_y\bar{\kappa} \right) dx$$

Virtual works – unit force theorem

virtual work of a unit force on the direction of the force



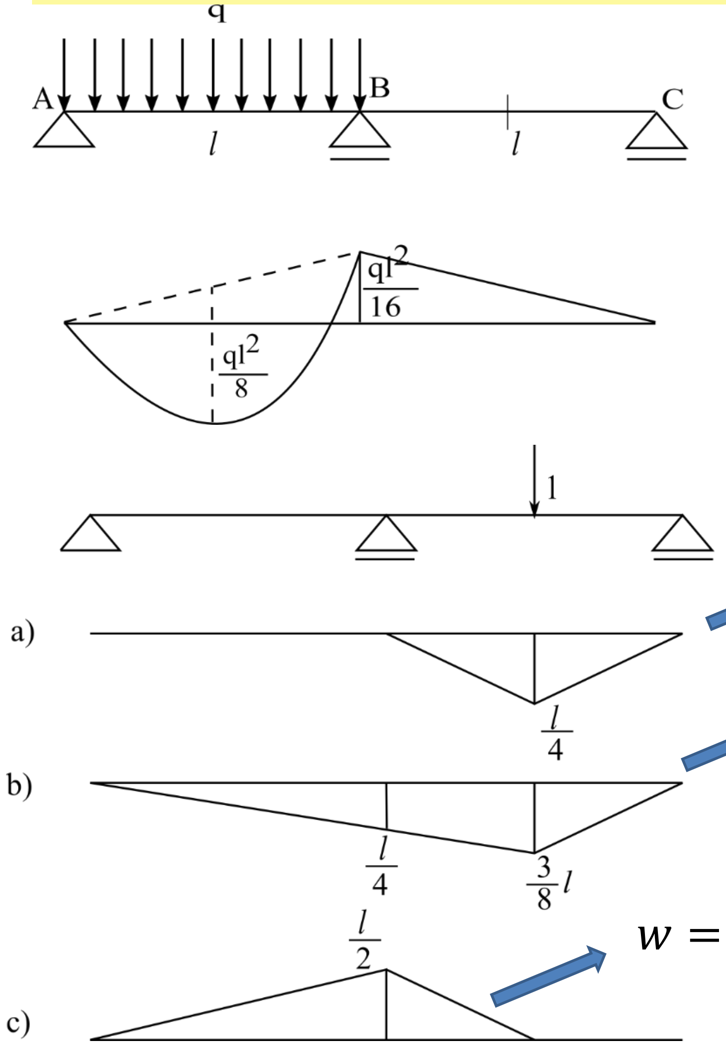
$$\bar{1}u_A = \int_L (\bar{N}_1\varepsilon + \bar{Q}_1\beta + \bar{M}_1\kappa)\, dx$$

usually the displacements are produced by bending, the influence of other cross-sectional forces can be neglected

$$\kappa = \frac{M}{EI} \qquad \rightarrow \qquad \bar{1}u_A = \int_L \frac{\bar{M}_1 M}{EI} dx$$

statically indeterminate structures: two theorems of reduction
1st – for real bending moments and virtual bending moments of fundamental statically determined state
2nd – for virtual bending moments and real bending moments of fundamental statically determined state

Virtual work application – an example



Determine the displacement at the middle of the span BC of the beam in Figure; the flexural stiffness is constant. The diagram of the bending moments is given.

The virtual work of the redundant reactions on their displacements, all are zero, is equal to zero.

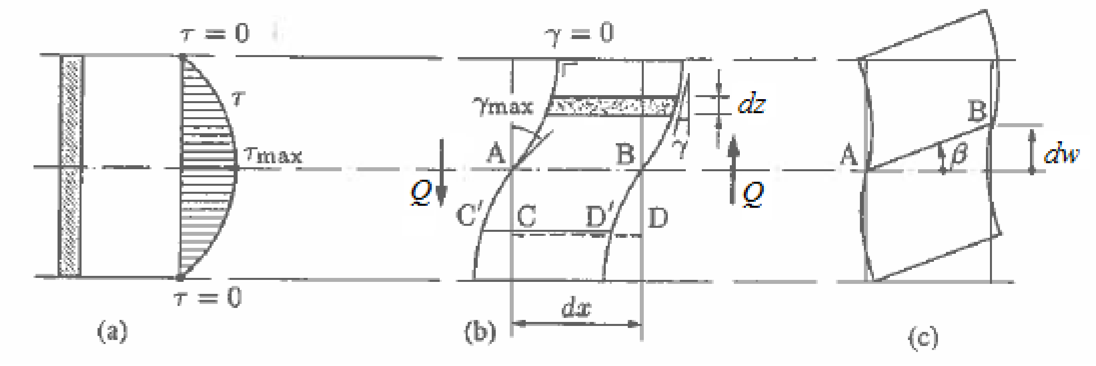
Assuming each of reaction as redundant, we get three schemes of virtual bending moments.

$$w = -\frac{1}{2}\cdot\frac{l}{4}\cdot\frac{l}{2}\cdot\left(\frac{1}{2}+\frac{1}{3}\cdot\frac{1}{2}\right)\frac{ql^2}{16EI} - \frac{1}{2}\cdot\frac{l}{4}\cdot\frac{l}{2}\cdot\frac{2}{3}\cdot\frac{1}{2}\cdot\frac{ql^2}{16EI} = -\frac{ql^4}{256EI}$$

$$\mathrm{w}=\frac{2}{3}\cdot\frac{ql^2}{8EI}\cdot l\cdot\frac{l}{2}\cdot\frac{l}{4} - \frac{1}{2}\cdot\frac{ql^2}{16EI}\cdot l\cdot\frac{2}{3}\cdot\frac{l}{4} - \frac{1}{2}\cdot\frac{ql^2}{16EI}\cdot\frac{\mathrm{l}}{2}\cdot\left(\frac{2}{3}\cdot\frac{l}{4}+\frac{1}{3}\cdot\frac{3}{8}\cdot l\right) - \frac{1}{2}\cdot\frac{ql^2}{32EI}\cdot\frac{l}{2}\cdot\left(\frac{2}{3}\cdot\frac{3}{8}\cdot l+\frac{1}{3}\cdot\frac{l}{4}\right) - \frac{1}{2}\cdot\frac{ql^2}{32EI}\cdot\frac{l}{2}\cdot\frac{2}{3}\cdot\frac{3}{8}\cdot l = -\frac{ql^4}{256EI}$$

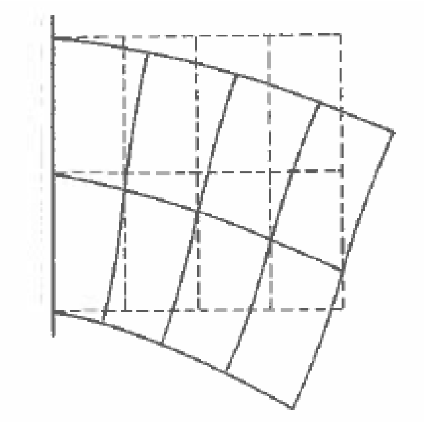
$$w = \frac{ql^2}{16EI}\cdot\frac{1}{2}\cdot l\cdot\frac{2}{3}\cdot\frac{l}{2} - \frac{2}{3}\cdot\frac{ql^2}{8EI}\cdot l\cdot\frac{l}{4} + \frac{l}{2}\cdot\frac{l}{2}\cdot\frac{1}{2}\cdot\frac{ql^2}{16EI}\cdot\left(\frac{1}{2}+\frac{2}{3}\cdot\frac{1}{2}\right) = -\frac{ql^4}{256EI}$$

Beams deflections – shear impact



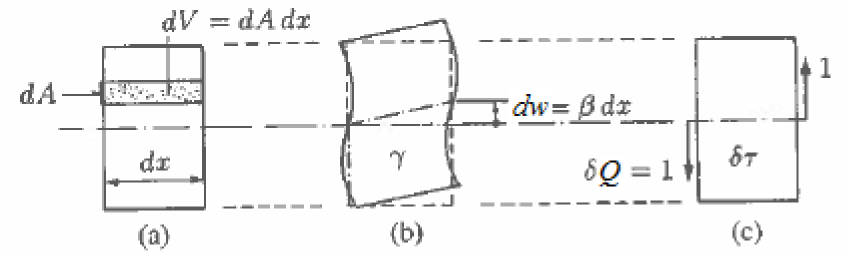
a) cross-section and stress distribution
b) distortion of sections
c) angle of average slip

the average slip angle cannot be easily calculated - the deformations deduced from the shear stress lead to kinematic incompatibility, see a cantilever below



conclusion: the deformation due to shear cannot be deduced independently of the bending moments

$$\gamma = \frac{\tau}{G}; \quad dw = \beta dx$$



numerical solution to the cantilever: constant shear force but cross-section distortions vary

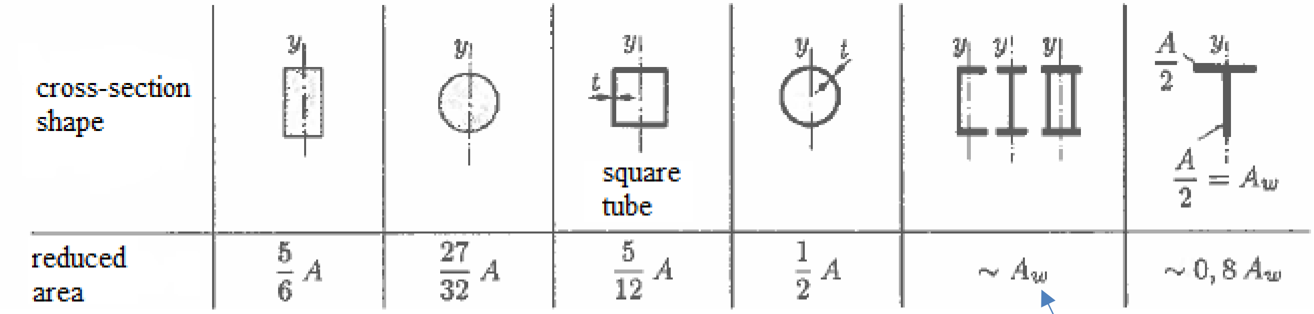
a) infinitesimal element, b) real kinematics, c) virtual statics

Shear impact on deflections – cont.

virtual works: $1dw = dw = \int_V \delta\sigma_{ij}\varepsilon_{ij}dV = \left(\int_A \delta\tau \cdot \gamma\, dA\right)dx \rightarrow \frac{dw}{dx} = \beta = \frac{1}{G}\int_A \delta\tau \cdot \tau dA$

$$Q = 1 \rightarrow \delta\tau = \tau_1;\ \tau = Q\tau_1 \rightarrow \frac{dw}{dx} = \beta = \frac{Q}{G}\int_A \tau_1^2 dA = \frac{Q}{G}\int_A \frac{S^2}{I^2 b^2}dA = \frac{Q}{GA}\frac{A}{I^2}\int_A \frac{S^2}{b^2}dA = \frac{Q}{GA}\mu = \frac{\mathrm{Q}}{\mathrm{GA_r}}$$

$\mu \stackrel{\text{def}}{=} \frac{A}{I^2}\int_A \frac{S^2}{b^2}dA$ $\quad\mu -$ energy shear coefficient (pure geometry); $A_r \stackrel{\text{def}}{=} \frac{A}{\mu} -$ reduced area



cross-section shape			square tube			$\frac{A}{2}$; $\frac{A}{2} = A_w$
reduced area	$\frac{5}{6}A$	$\frac{27}{32}A$	$\frac{5}{12}A$	$\frac{1}{2}A$	$\sim A_w$	$\sim 0{,}8\,A_w$

web area

Shear impact on deflections – cont.

prismatic homogeneous bar: $GA_r = \text{const}$

$$\frac{dw}{dx} = \beta = \frac{Q}{GA_r} \rightarrow \frac{d^2w}{dx^2} = \frac{1}{BA_r} \cdot \frac{dQ}{dx} = -\frac{q}{GA_r} = -\frac{1}{GA_r}\frac{d^2M}{dx^2}$$

curvature due to shear action

$$w_{\text{total}} = w_{\text{bending}} + w_{\text{shear}}$$

but the integration of bending-dependent and shear-dependent equations should be performed separately because of different nature of the boundary conditions

for simply supported one-span beam numerical values of shear and bending deflections ratios:

h/L		1/20	1/15	1/10	1/8
w_q/w_M (%)	rectangular section	0.8	1.3	3.0	4.7
	web-flange profile	22	30	45	56

here values for maximal stress in web, usually, these values are 3-5 times smaller

Final remarks: the beams displacements due to the shear force are negligible.

That’s all folks!