Strength of Materials

7. Torsion

Problem formulation

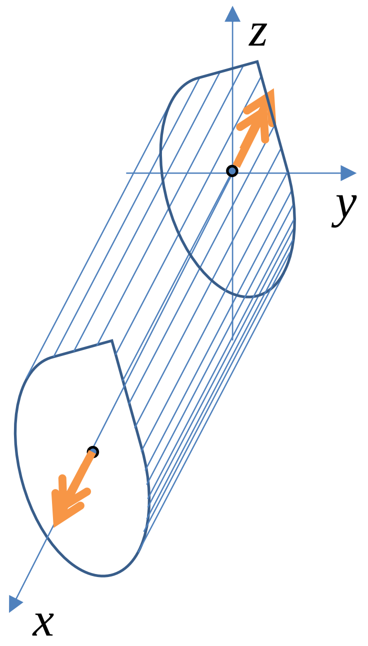
straight, prismatic bar with bottom surfaces perpendicular to the bar axis, cross-section of arbitrary shape

distributed loads on bottom that yields torque as only cross-sectional force, side surface free of loads, no volume forces (neglected)

(with use of de Saint-Venant principle the distributed loads can be replaced by a torque)

kinematic boundary conditions: bar fixed at one point (all displacements and their derivatives vanish there), the rest part of the bottom can deform freely

(this is the case of not constricted torsion in contrast to a constricted torsion where the whole bottom surface is fixed)



come back to BVP

NE (Navier)

$$\sigma_{ij,j} + P_i = 0$$

$$q_i = \sigma_{ij} n_j \quad \text{SBC}$$

CE (Cauchy)

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

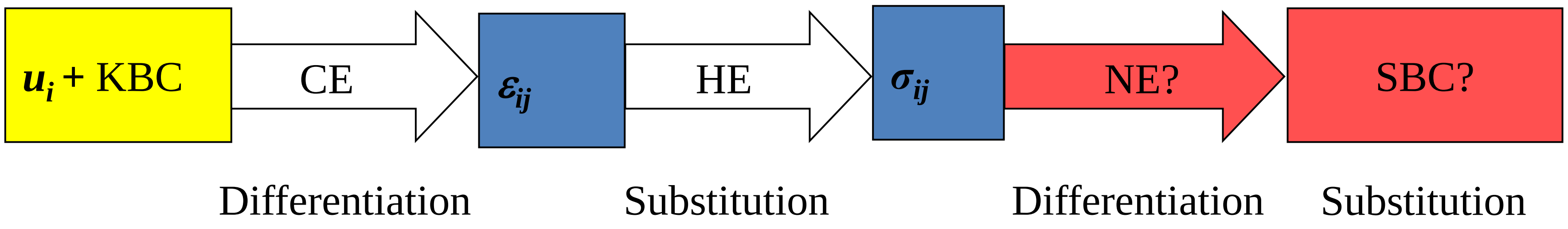
$$u_i\Big|_{S_u} = \ldots \quad \text{KBC}$$

HE (Hooke)

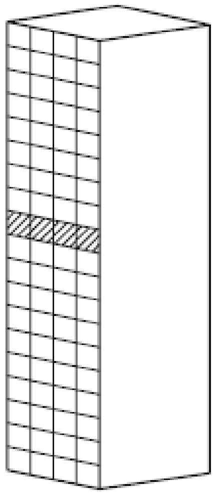
$$\sigma_{ij} = 2G\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij}$$

Semi-inversed methods

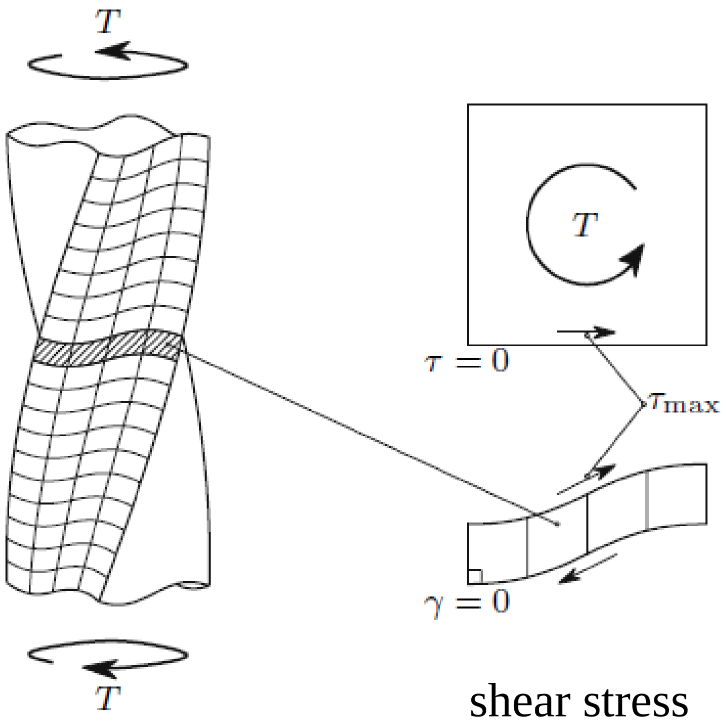
Kinematic approach: 3 functions $\boldsymbol{u_i}$ satisfying KBC are proposed, and then the strains are found by differentiation according to CE, and inserted into algebraic HE to obtain stresses which have to satisfy NE and SBC



Definitions

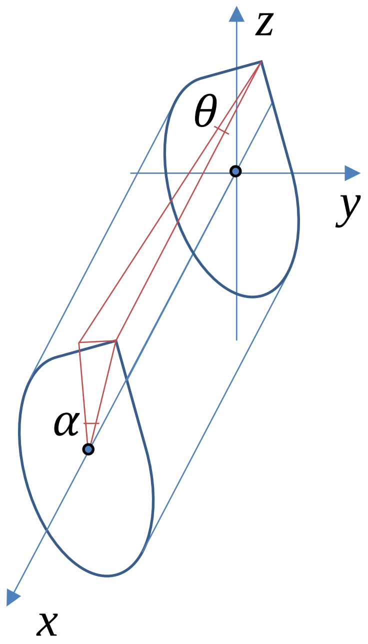


prism before loading



prism after loading

shear stress



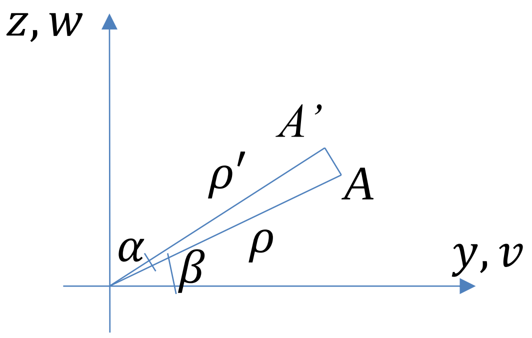
α – torsion (twist) angle, in [rd]
θ – unit torsion angle, in [rd/m]

distortion (warping) – a change of the cross-section plane into some deformed surface

de Saint-Venant kinematic approach

de Saint-Venant assumptions:

- the projection of the cross-section onto its initial plane remains unchanged (as the rigid membrane)
- the cross-section distortion consists in the movement “out-of-plane” only and is the same for all cross-sections



the displacement components can be written:

$$\rho \approx \rho', \qquad \sin\alpha \approx \alpha, \qquad \cos\alpha \approx 1, \qquad \alpha = \theta x$$

$$v = -\rho\alpha\sin\beta = -\theta xz, \qquad w = \rho\alpha\cos\beta = \theta xy,$$

distortion function: $u = \theta\varphi(y,z)$ (no dependence on x)

SBC: $v(0,0,0) = w(0,0,0) = 0$, ($u(0,0,0) = 0$ not yet satisf.)

similarly: $v_{,x}(0,0,0) = v_{,z}(0,0,0) = w_{,x}(0,0,0) = w_{,y}(0,0,0) = 0$

(and $u_{,y}(0,0,0) = u_{,z}(0,0,0) = 0$ not yet satisfied)

Boundary value problem solution

$u = \theta\varphi(y,z)$, $\quad \varphi$ − distortion function
$v = -\theta xz$,
$w = \theta xy$

strains

$$\varepsilon_x = \frac{\partial u}{\partial x} = 0$$
$$\varepsilon_y = \frac{\partial v}{\partial y} = 0$$
$$\varepsilon_z = \frac{\partial w}{\partial z} = 0$$
$$\varepsilon_{xy} = \frac{1}{2}\left(\theta\frac{\partial\varphi}{\partial y} - \theta z\right)$$
$$\varepsilon_{xz} = \frac{1}{2}\left(\theta\frac{\partial\varphi}{\partial z} + \theta y\right)$$
$$\varepsilon_{yz} = \frac{1}{2}(-\theta x + \theta x) = 0$$

the strain matrix $T_\varepsilon = \begin{bmatrix} 0 & \frac{1}{2}\theta\left(\frac{\partial\varphi}{\partial y} - z\right) & \frac{1}{2}\theta\left(\frac{\partial\varphi}{\partial z} + y\right) \\ & 0 & 0 \\ & & 0 \end{bmatrix}$

the stress matrix $T_\sigma = \begin{bmatrix} 0 & G\theta\left(\frac{\partial\varphi}{\partial y} - z\right) & G\theta\left(\frac{\partial\varphi}{\partial z} + y\right) \\ & 0 & 0 \\ & & 0 \end{bmatrix}$

Navier equations
the first:

$$\frac{1}{2}G\theta\frac{\partial^2\varphi}{\partial y^2} + \frac{1}{2}G\theta\frac{\partial^2\varphi}{\partial z^2} = 0 \rightarrow \nabla^2\varphi = 0 \qquad (1)$$

Laplace equation – φ is the harmonic function

another notation: $\Delta\varphi = 0,\ \Delta$ − Laplacian

the second and the third are fulfilled because the distortion function doesn't depend on *x*

BVP solution – cont.

SBC

on the side surface, $\vec{n}(0,m,n)$:

$\left(\frac{\partial\varphi}{\partial y} - z\right)m + \left(\frac{\partial\varphi}{\partial z} + y\right)n = 0,$ (the second and the third are identically fulfilled) (2)

on the bottoms, $\vec{n}(\pm 1,0,0)$:

$q_y = G\theta\left(\frac{\partial\varphi}{\partial y} - z\right)(\pm 1), q_z = G\theta\left(\frac{\partial\varphi}{\partial z} + y\right)(\pm 1),$ (the first equation is identically fulfilled) (3)

KBC

$\varphi(0,0,0) = 0, \varphi_{,y}(0,0,0) = \varphi_{,z}(0,0,0) = 0$ (4)

Neumann problem:

the equation (1) with the boundary conditions (2); single solution exists accurate to within a constant

so, we can fulfill the condition 4_1

if the cross-section have at least one symmetry axis, the conditions $4_2, 4_3$ will be fulfilled, and also, for other shape of the cross-section with good precision

only the conditions (3) remain, thus we admit the loading in the form:

$$q_y = G\theta\left(\frac{\partial\varphi}{\partial y} - z\right)(\pm 1), q_z = G\theta\left(\frac{\partial\varphi}{\partial z} + y\right)(\pm 1)$$

BVP solution – final remarks

statically equivalent torque

$$M_x = \iint_A (q_z y - q_y z) dA = \iint_A \left[G\theta \left(\frac{\partial \varphi}{\partial z} + y \right) y - G\theta \left(\frac{\partial \varphi}{\partial y} - z \right) z \right] dA = G\theta \iint_A \left(\frac{\partial \varphi}{\partial z} y - \frac{\partial \varphi}{\partial y} z + y^2 + z^2 \right) dA$$

torsion inertia moment, I_x

$$I_x \stackrel{\text{def}}{=} \iint_A \left(\frac{\partial \varphi}{\partial z} y - \frac{\partial \varphi}{\partial y} z + y^2 + z^2 \right) dA, \quad [\mathrm{m}^4]$$

finally:

$$M_x = \theta G I_x$$

$$\theta = \frac{M_x}{G I_x}$$

$G I_x$ – torsional stiffness

Solid circular shaft

for the circular cross-section, the static boundary conditions on the side surface $\vec{n}(0, \cos\beta, \sin\beta)$:

$\frac{\partial\varphi}{\partial y}y + \frac{\partial\varphi}{\partial z}z = 0$ and $\varphi(0,0,0) = 0$

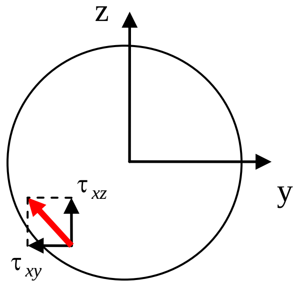
because of the homogeneity of the equations, i.e. Laplace equation and the boundary conditions, the distortion function vanishes:

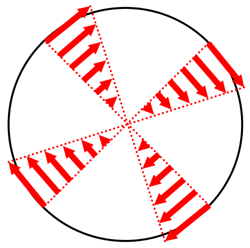
$$\varphi(y,z) \equiv 0$$

The torsion inertia moment becomes the polar inertia moment $I_x = \iint_A (y^2 + z^2)dA = I_0$

the shear stresses

$$\tau_{xy} = -\frac{M_x}{I_0}z, \qquad \tau_{xz} = \frac{M_x}{I_0}y$$





the stress vector is perpendicular to the radius and its length is

$$\tau = \sqrt{\tau_{xy}^2 + \tau_{xz}^2} = \frac{M_x}{I_0}\sqrt{y^2 + z^2} = \frac{M_x}{I_0}r$$

$$\max\tau = \frac{M_x}{I_0}R = \frac{M_x}{W_0} \qquad W_0 \overset{\text{def}}{=} \frac{I_0}{R}$$ torsional (twist) section factor

$$I_0 = \frac{\pi d^4}{32} = \frac{\pi r^4}{2}, \qquad W_0 = \frac{\pi d^3}{16} = \frac{\pi r^3}{2}$$

Torsion of circular shaft – design

The carrying capacity limit state $\max\tau = \dfrac{M_x}{W_0} \leq R_t$ $\quad R_t -$ shear stress calculation limit

The usability limit state $\theta = \frac{M_x}{GI_0} \leq \theta_{\text{allowable}}, \quad \left[\dfrac{\text{rd}}{\text{m}} = \dfrac{1}{\text{m}}\right]$ (not in degrees!)

or

if $M_x = \text{const}$: $\alpha = \theta l = \frac{M_x l}{GI_0} \leq \alpha_{\text{allowable}}, \; [rd = 1]$

if $M_x \neq \text{const}$: $\alpha = \int_l \theta dx = \int_l \frac{M_x}{GI_0} dx$

Transmission shafts

transmitted power: $P = M_x\omega = M_x 2\pi f,$

$M_x -$ torque, $\omega -$ (angular) velocity $\left[\dfrac{\text{rd}}{\text{s}}\right]$, $f -$ rotation frequency (a number of revolutions per second)

high-speed/low-speed engines

Hollow shafts

inertia moment $I_0 = \frac{\pi D^4}{32} - \frac{\pi d^4}{32} = \frac{\pi}{32}(D^4 - d^4)$

cross-section torsion factor $W_0 = \frac{I_0}{D/2} = \frac{\pi}{16D}(D^4 - d^4)$

Prandtl function – another solution

homogeneous Laplace equation and non-homogeneous boundary conditions – the Neumann problem
nonhomogeneous Poisson equation and homogeneous boundary conditions – the Dirichlet problem

Prandtl function, ψ :

$$\tau_{xy} = G\theta \frac{\partial \psi}{\partial z}, \tau_{xz} = -G\theta \frac{\partial \psi}{\partial y}$$

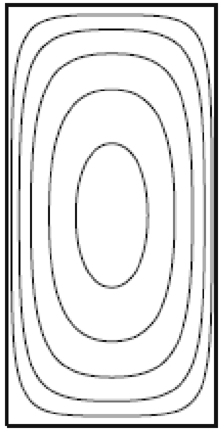
$$\frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = -2$$

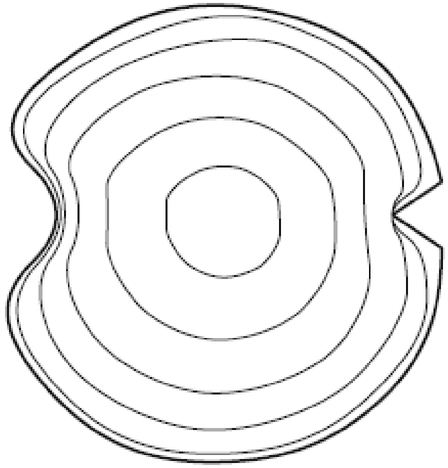
on the contour ψ=const, usually 0 is admitted

$$M_x = 2G\theta \iint_A \psi dA$$

Prandtl's analogy, soap bubble (membrane) analogy, Greenhill's hydrodynamic analogy

Prandtl's analogy



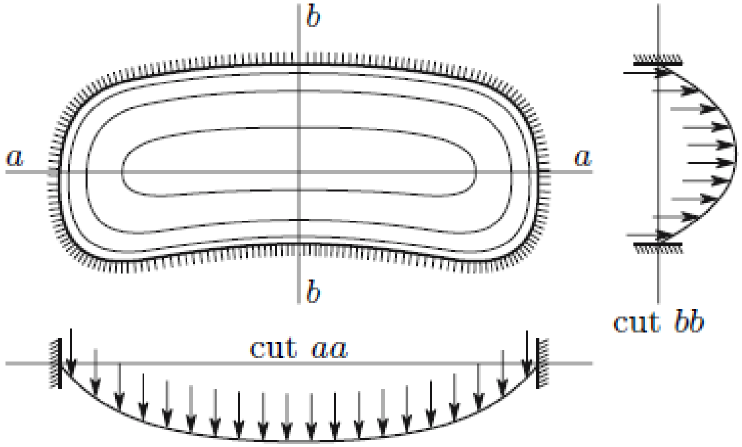


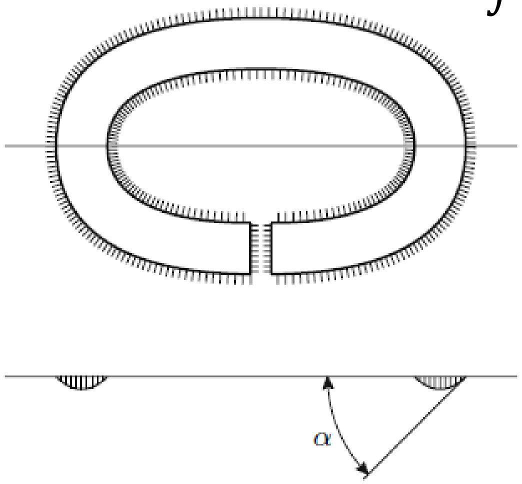
the volume of the solid between the Prandtl's function and the cross-section plane is proportional to the torque

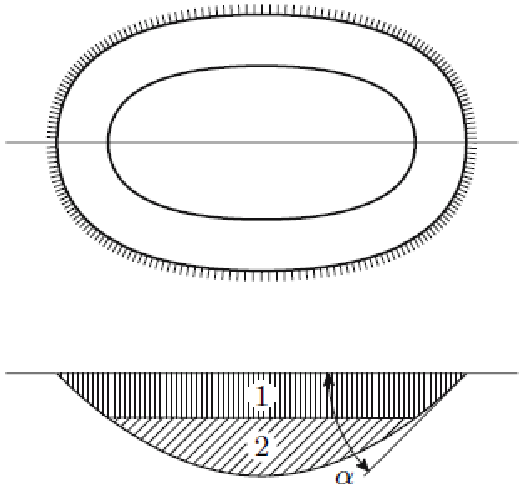
the slope is proportional to the resultant shear stress τ

for multiply connected cross-sections the additional conditions of displacements compatibility is required, and leads to the conditions that circulation of the shear stress over the hole contour is proportional to the hole area:

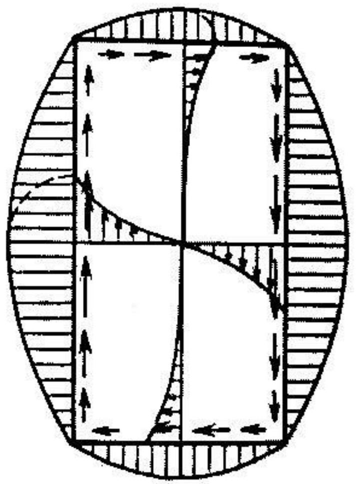
$$\oint \tau ds = 2G\theta A$$







Rectangular cross-section



trigonometric series expansion for the distortion function $\varphi = \sum_k^{1,3,5\ldots} f_k(y) \cos\frac{k\pi y}{a}$

and right side of the Poison equation in Fourier series $-2G\theta = \sum_k^{1,3,5,\ldots} A_k \cos\frac{k\pi y}{a}$

we get differential equation $f_k''(y) - \left(\frac{k\pi}{a}\right)^2 f_k(y) = A_k$ with the solution

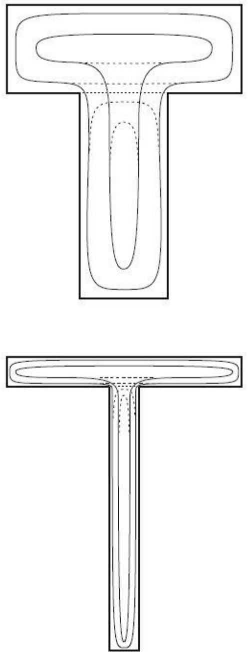
$$\varphi = G\theta\left[\left(\frac{a}{2}\right)^2 - y^2\right] - \frac{8G\theta a^2}{\pi^3}\sum_k^{1,3,5,\ldots}\frac{(-1)^{\frac{k-1}{2}}}{k^3}\cdot\frac{\cosh\frac{k\pi z}{a}}{\cosh\frac{k\pi b}{2a}}\cos\frac{k\pi y}{a}$$

tabulated solutions: $W_x \stackrel{\text{def}}{=} \alpha b^2 h; I_x \stackrel{\text{def}}{=} \beta b^3 h, \left\{\theta = \frac{M_x}{GI_x}, \tau_{\max} = \frac{M_x}{W_x}, \gamma = \tau\left(\frac{b}{2}\right)/\tau_{\max}\right\}$

h/b	1	1.5	2	3	4	6	8	10	∞
α	0.208	0.231	0.246	0.267	0.282	0.299	0.307	0.312	1/3
β	0.141	0.196	0.229	0.263	0.281	0.299	0.307	0.312	1/3
γ	1	0.859	0.795	0.753	0.745	0.743	0.742	0.742	

Thin-walled profiles

developable profiles (C-channels, Z-sections, angle sections, etc.)
may be substituted by a rectangle with the same area and a height equal to the length of the middle line
example: C-channel profile 300×10 (web) and 90×16 (flanges) → a rectangle $h = 2 \times 85 + 284 = 454$ mm and average width $b = 12.25$ mm

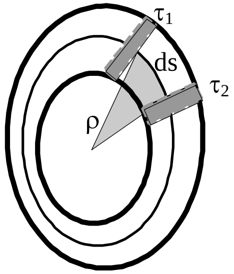


not developable profiles (I-beams, W-beams, T-beams, etc., the middle-line bifurcates)
may be cut into simple rectangles, provided that:

- $\sum M_{xi} = M_x$
- $\theta_i = idem \rightarrow \frac{M_{xi}}{\mathrm{GI_{xi}}} = \frac{\mathrm{M_{x(i+1)}}}{\mathrm{GI_{x(i+1)}}}, \mathrm{i} = 1, \cdots, n-1$

hence M_{xi} and the common unit torsion angle θ of every rectangle and the whole section
the smaller is the connection zone, the smaller will be an error
if the thickness is constant, we can use one rectangle instead

hollow structural sections
from the hydrodynamic analogy: $\tau_1 \delta_1 = \tau_2 \delta_2 = \mathrm{const}$
Bredt's first formula: $\tau_{\max} = \frac{M_x}{2A\delta_{\min}}$
Bredt's second formula: $I_x = \frac{4A^2}{\oint_c \frac{ds}{\delta}}, \delta = \mathrm{const} \rightarrow I_x = \frac{4A^2\delta}{c}$



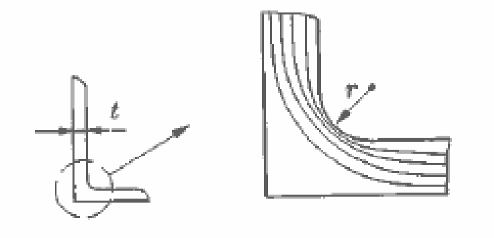
Applicability of Bredt's formula

comparison with exact solution to tubular cross-section (instead of linear stress distribution, the Bredt's formula contains constant, rectangular distribution)

$$\beta = \frac{\tau_{\max}^{ex}}{\tau_{\max}^{B}} = \frac{4+2\alpha}{4+\alpha^2}, \quad \gamma = \frac{\theta^{ex}}{\theta^{B}} = \frac{4}{4+\alpha^2}, \quad \alpha = \frac{\delta}{r_m}$$

α	0	0.05	0.1	0.15	0.25	0.5	1
β	1.0000	1.0244	1.0474	1.0690	1.1077	1.1765	1.2000
γ	1.0000	0.9994	0.9975	0.9944	0.9846	0.9412	0.8000

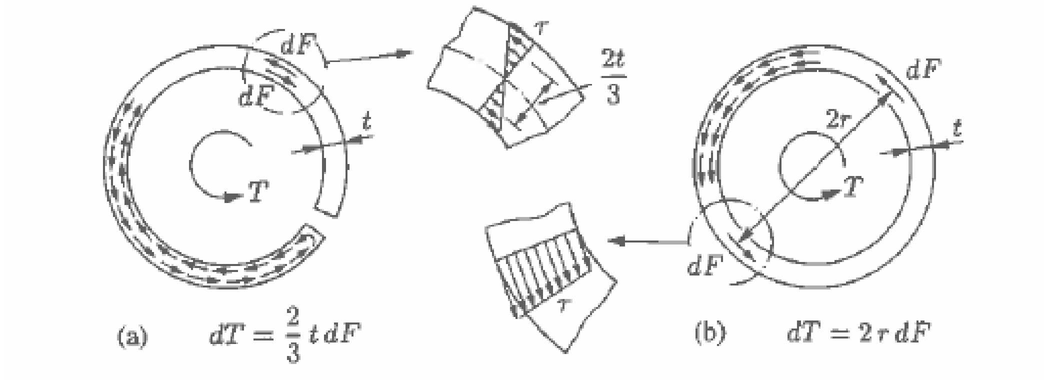
stress concentration inside the angles, in the round off; greater round offs produce greater twisting inertia moment (up to 25%)



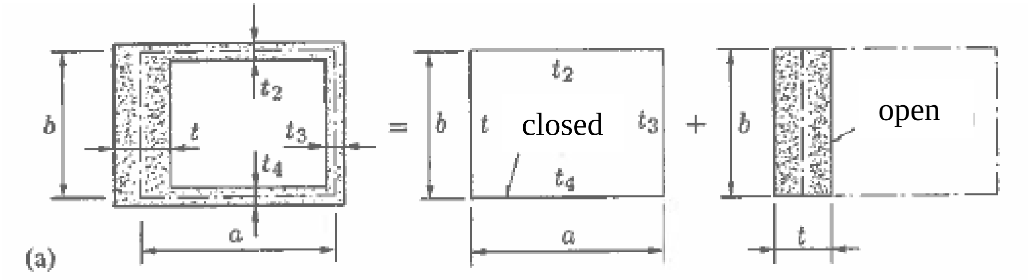
r/t	2.0	1.0	0.5	0.3
$\tau_r/\tau_{\max}$	~1.0	~1.4	~1.8	~2.1

Torsion - complements

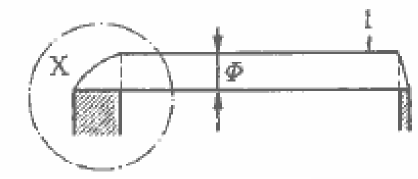
The bearing capacity of thin-walled closed profile is much greater than those with similar dimensions but open for the ring $\frac{\delta}{r} = 0.1 \rightarrow$ the torsional stiffness of the open section is about 260 times smaller and the bearing capacity is about 26 times smaller when compared with the closed section

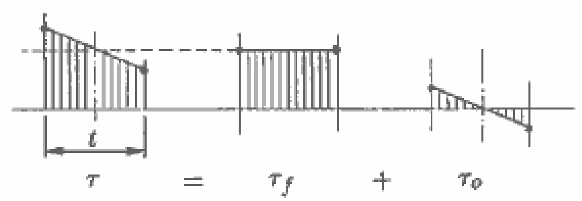


correction of the Bredt’s formulas by breaking cross-section into closed and open parts



revolved section of the Prandtl’s function:

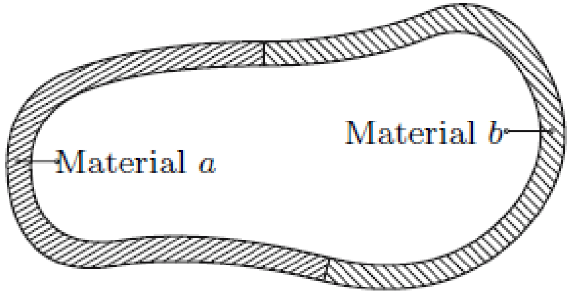




Torsion of composite section

The Bredt's formulas can be generalized to composite beams made of two linear elastic materials

(a) tubular cross-section with homogeneous wall



Comparing the work of the internal forces with the work of the external forces, we get

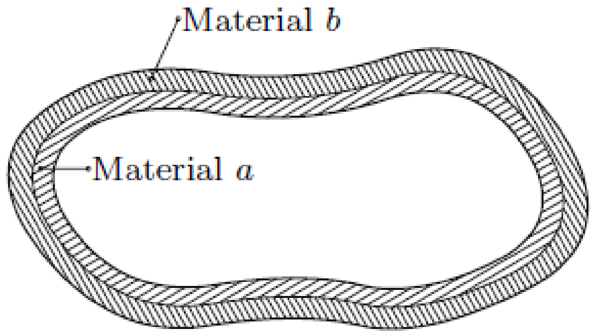
$$\frac{1}{2}\int_c \tau\gamma dc = \frac{1}{2}\int_c \frac{\tau^2}{G} dc = \frac{1}{2}\left(\int_c \frac{\tau^2}{G_a}\delta dc + \int_c \frac{\tau^2}{G_b}\delta dc\right) = \frac{1}{2}\frac{M_x^2}{4A^2}\left(\frac{1}{G_a}\int_a \frac{dc}{\delta} + \frac{1}{G_b}\int_b \frac{dc}{\delta}\right) = \frac{1}{2}M_x\theta$$

From this comparison we get:

$$\theta = \frac{M_x}{4A^2}\left(\frac{1}{G_a}\int_a \frac{dc}{\delta} + \frac{1}{G_b}\int_b \frac{dc}{\delta}\right) \rightarrow \ \theta = \underbrace{\frac{M_x}{4A^2 G_a}\left(\int_a \frac{dc}{\delta} + \int_b \frac{G_a}{G_b}\frac{dc}{\delta}\right)}_{\text{homogenization in material } a} = \underbrace{\frac{M_x}{4A^2 G_b}\left(\int_a \frac{G_b}{G_a}\frac{dc}{\delta} + \int_b \frac{dc}{\delta}\right)}_{\text{homogenization in material } b}$$

Torsion of composite section

(b) tubular cross-section with composite wall



The Bredt's formula is valid only in terms of stress flow: $f = \tau_a \delta_a + \tau_b \delta_b = \frac{M_x}{2A}$, A – area limited by the center line of the composite section. As the distortion must be the same in the two materials, we have:

$$\gamma = \frac{\tau_a}{G_a} = \frac{\tau_b}{G_b} \rightarrow \ \tau_b = \frac{G_b}{G_a} \tau_a \rightarrow \ \tau_a = G_a \frac{M_x}{2A(G_a \delta_a + G_b \delta_b)}, \tau_b = G_b \frac{M_x}{2A(G_a \delta_a + G_b \delta_b)}$$

comparing the works $\frac{1}{2} \oint \left(\frac{\tau_a^2}{G_a} \delta_a + \frac{\tau_b^2}{G_b} \delta_b \right) dc = \frac{1}{2} M_x \theta \rightarrow \ \theta = \frac{M_x}{4A^2} \oint \frac{G_a \delta_a + G_b \delta_b}{(G_a \delta_a + G_b \delta_b)^2} dc = \frac{M_x}{4A^2} \oint \frac{dc}{G_a \delta_a + G_b \delta_b}$

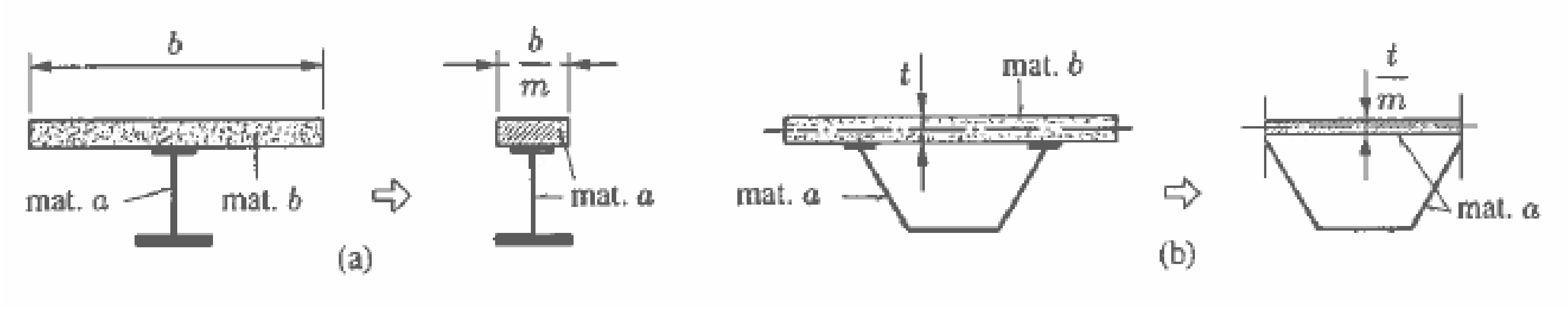
and, using the homogenization concept to the expression of unit twist angle:

$$\theta = \underbrace{\frac{M_x}{4A^2 G_a} \oint \frac{dc}{\delta_a + \frac{G_a}{G_b} \delta_b}}_{\text{homogenization in material } a} = \underbrace{\frac{M_x}{4A^2 G_b} \oint \frac{dc}{\frac{G_a}{G_b} \delta_a + \delta_b}}_{\text{homogenization in material } b}$$

Torsion of composite section

the equivalence ratio: $m = \frac{G_a}{G_b}$

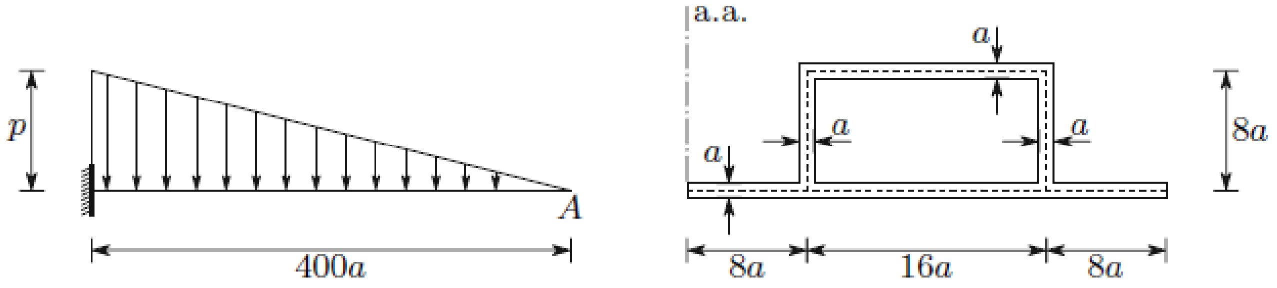
similar procedure as for bending of composite beam – we introduce an equivalent cross-section:
for open cross-section we divide by *m* the length of rectangle *b*
for closed cross-section we divide the thickness of the wall by *m*



Equivalent cross-sections for the composite beams made from materials *a* and *b*
a) open section, b) closed section (flange overhangs neglected)

Torsion - examples

(da Silva) The beam in the Figure below is loaded in the plane shown as the action axis (a.a.). Determine the rotation of cross-section *A* around the beam's axis.



Solution

the load per unit length: $q(x) = \frac{p}{400a}x$

the shear force: $Q(x) = \int q dx = \frac{p}{800a}x^2$

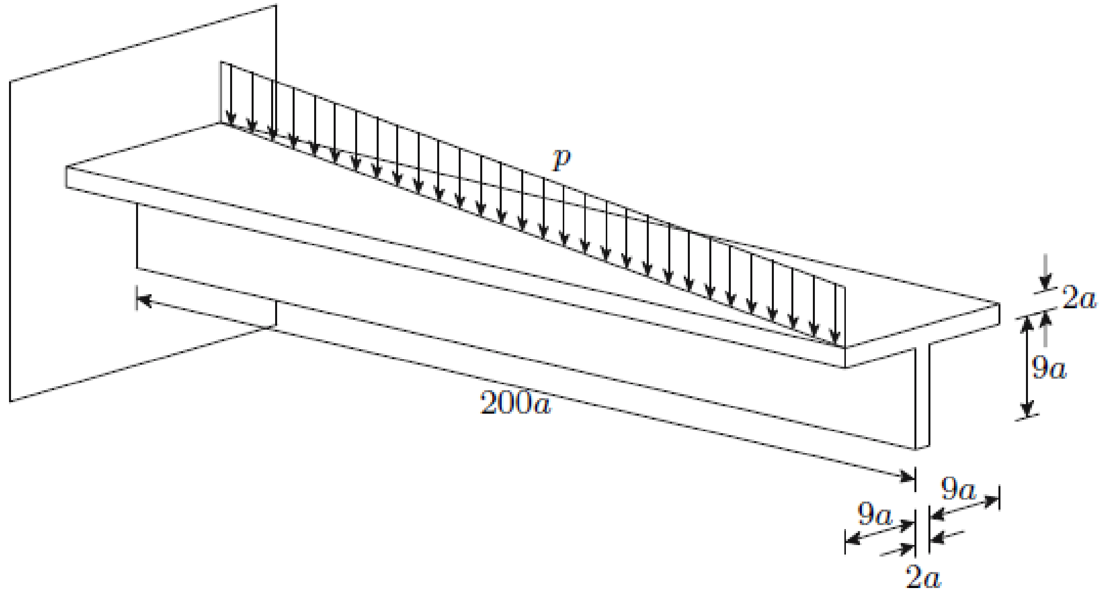
the twisting moment (against the shear center): $M_x(x) = \frac{p}{800a}x^2 \cdot 16a = \frac{p}{50}x^2$

the twist inertia moment (using the Bredt's second formula): $I_x = \frac{4A^2\delta}{c} = \frac{4(16a \cdot 8a)^2 a}{48a} = 1365a^4$

the twist angle: $\alpha = \int_0^l \theta dx = \int_0^l \frac{M_x(x)}{GI_x} dx = \frac{1}{GI_x} \cdot \frac{p}{150}(400a)^3 = 312.6\frac{p}{Ga}$

Torsion – example

(da Silva) The cantilever beam in the Figure below is made of a material with a shear modulus G. Determine the rotation of the right cross-section around the bar axis, caused by uniformly distributed loading p.



Solution

the shear force: $Q(x) = p\frac{x}{\left(\frac{200}{\sqrt{200^2+20^2}}\right)} = 1.005px$

the lever of shear force (vanishes at the fixed end):

$$d(x) = 10a - \frac{x}{20}$$

the twist inertia moment (as one rectangle):

$$I_x = \frac{1}{3}(20a + 9a)\cdot(2a)^3 = \frac{232}{3}a^4$$

the rotation angle of the free end:

$$\alpha = \frac{1}{GI_x}\int_0^l Q dx = \frac{1}{GI_x}\int_0^{200a} 1.005px\left(10a - \frac{x}{20}\right)dx = \frac{25000\cdot 1.005}{29}\cdot\frac{p}{Ga} \approx 866\cdot\frac{p}{Ga}$$

It’s over, thank you!