Strength of Materials

8. Energy

Thermodynamics – definitions

A thermodynamic system can be:

- closed (isolated), exchanges neither matter nor energy (ex.: the Universe viz. cosmos)
- semi-permeable, exchanges only energy (ex.: flatiron)
- permeable (open), exchanges matter and energy with surroundings (ex.: making jam)

The system is enclosed by walls, fixed or movable:

- adiabatic (ideal thermal isolation, no energy exchange through it, ex.: vacuum bottle)
- diathermal (ideal permeability for temperature exchange – zeroth law of thermodynamics and the temperature measurement)

The first law of thermodynamics – two postulates

1) The principle of energy conservation: total energy quantity is constant, $dE = 0$

The total energy is composed from:

- potential energy (resulting from externally imposed force field, like gravity)
- kinetic energy (resulting from the system motion as a whole)
- the remainder of energy constitutes internal energy (elastic, irradiation, chemical, thermal, magnetic and many others)

$$E = E_p + E_k + W$$

Energy – the first law of thermodynamics

2) The internal energy can be exchanged in two ways: by work or by heat

$$dW = \delta L + \delta Q$$

δ – diminutives (in general not a differential)
this means that the internal energy is an exact differential and that it doesn't depend on the path (only actual and initial states count)
and that work and heat depend on the path (not only on the initial and actual states)

For an adiabatic process (without heat exchange and production):

$$dW = \delta L$$

The principle of virtual works

$$\delta W_{int} = \delta W_{ext}$$

$$\int_A q_i \delta u_i dA = \int_{A_\sigma} q_i \delta u_i dA + \int_{A_u} q_i \delta u_i dA = \int_{A_\sigma} q_i \delta u_i dA$$

$$\int_V \sigma_{ij} \delta \varepsilon_{ij} dV = \int_V b_i \delta u_i dV + \int_{A_\sigma} q_i \delta u_i dA$$

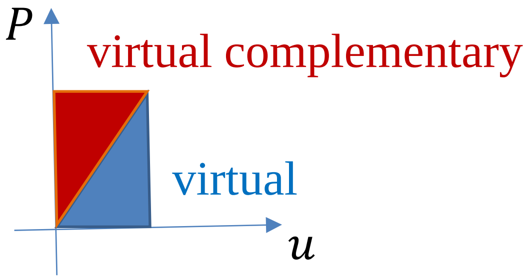
(internal + external)

Potential and complementary energy

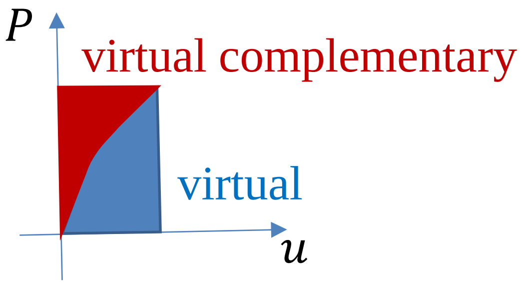
The principle of complementary virtual works

$$\int_V \delta\sigma_{ij}\varepsilon_{ij}dV = \int_V \delta b_i u_i dV + \int_A \delta q_i u_i dA \qquad \text{(internal + external)}$$

linear material



nonlinear material

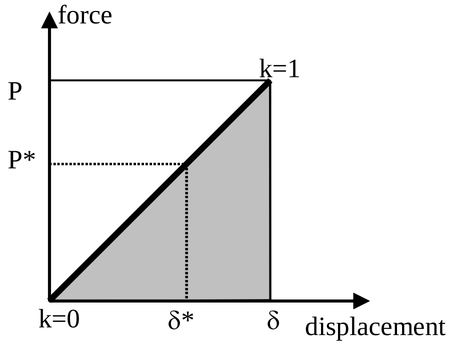


For a linearly elastic body, the path of the work is uniquely defined and the diminutive of work becomes an exact differential:

$$dW = dL$$

Clapeyron’s theorem:
The elastic energy of a body is equal to one half of the products of all generalized forces and respective generalized displacements:

$$L = \int_0^1 P\delta k dk = P\delta \int_0^1 k dk = \frac{1}{2}P\delta = W_{el}$$



Betti reciprocal work theorem

$$\int_V \sigma_{ij}\varepsilon'_{ij}\,dV = \int_V D_{ijkl}\varepsilon_{kl}\varepsilon'_{ij}\,dV \quad = \int_V D_{klij}\varepsilon'_{ij}\varepsilon_{kl}dV = \int_V \sigma'_{kl}\varepsilon_{kl}dV$$

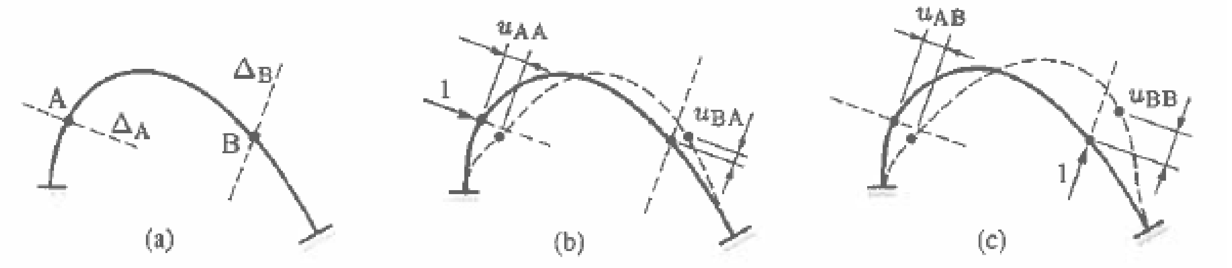
$$\int_V b_i u'_i\,dV + \int_{A_\sigma} q_i u'_i dA = \int_V b'_i u_i\,dV + \int_{A_\sigma} q'_i u_i dA$$

for a linear elastic structure subject to two sets of forces P_i and Q_i the work done by the set P through the displacements produced by the set Q is equal to the work done by the set Q through the displacements produced by the set P

$$\sum Pu' = \sum P'u$$

(influence lines, boundary element method)

Particular case: Maxwell’s theorem



$$1u_{AB} + 0u_{BB} = 0u_{AA} + 1u_{BA}$$

so:

$$\boxed{u_{AB} = u_{BA}}$$

Variational principles

when an exact solution of BVP is not known, we can seek an approximate solution yet
the idea of variational methods consists in looking up for integral terms, determined for a specific class function, with the stationary conditions equivalent with the solution to the BVP
we admit the static and kinematic fields as an independent variables set to characterize energies: potential and complementary
these functionals become ordinary functions of variables: kinematic and static
Lagrange principle of minimum total potential energy:

$$\delta[U - L] = 0$$

a structure deforms to a position that (locally) minimizes the total potential energy
Castigliano’s principle of minimum total complementary energy:

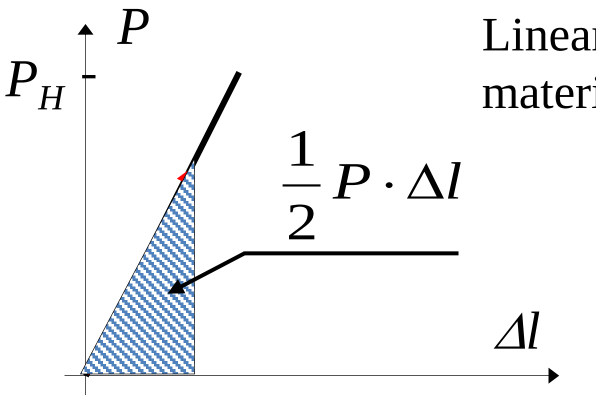
$$\delta[U_c - L_c] = 0$$

Castigliano’s theorem:

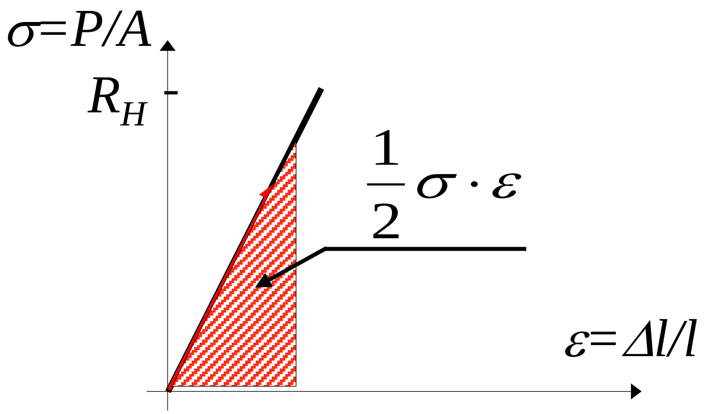
$$\frac{\partial U_c}{\partial F} = u$$

the partial derivative of the strain energy, considered as a function of the applied forces acting on a linearly elastic structure, with respect to one of these forces, is equal to the displacement in the direction of the force at its point of application

Elastic energy



Linear elastic material



External work = Internal work

$$L = \frac{1}{2} P \cdot \Delta l \quad = \quad W_p = \int_V \frac{1}{2} \sigma \cdot \varepsilon dV = \int_V \frac{1}{2} \sigma \cdot \frac{\sigma}{E} dV = \int_V \frac{\sigma^2}{2E} dV$$

For a prismatic bar: $\sigma = \frac{N}{A} = \frac{P}{A}$ $\quad V = A \cdot l$

$$W_p = \int_V \frac{N^2}{2A^2 E} dV = \frac{P^2}{2A^2 E} A \cdot l = \frac{P^2 l}{2AE}$$

$$\boxed{\Delta l = \frac{P \cdot l}{EA}}$$

Elastic Energy – cont.

$$\dot{L} = \int_V P_i \dot{u}_i dV + \int_S p_i \dot{u}_i dS = \int_V P_i \dot{u}_i dV + \int_S \sigma_{ij} n_j \dot{u}_i dS = \cdots$$

$$= \int_V P_i \dot{u}_i dV + \int_V \left(\sigma_{ij}\dot{u}_i\right)_{,j} dV = \int_V \left(P_i \dot{u}_i + \sigma_{ij,j}\dot{u}_i + \sigma_{ij}\dot{u}_{i,j}\right) dV = \cdots$$

$$= \int_V \left[\left(P_i + \sigma_{ij,j}\right)\dot{u}_i + \sigma_{ij}\dot{u}_{i,j}\right] dV = \int_V \sigma_{ij}\dot{\varepsilon}_{ij} dV = \boxed{\int_V T_\sigma T_{\dot{\varepsilon}} dV = \dot{L} = \dot{W}_p}$$

$$\dot{W}_p = \int_V (A_\sigma + D_\sigma)(A_{\dot{\varepsilon}} + D_{\dot{\varepsilon}}) dV = \int_V (A_\sigma A_{\dot{\varepsilon}} + D_\sigma D_{\dot{\varepsilon}} + A_\sigma D_{\dot{\varepsilon}} + D_\sigma A_{\dot{\varepsilon}}) dV$$

$$\mathrm{A}_\sigma \mathrm{D}_{\dot{\varepsilon}} = \sigma_m \delta_{ij}\left(\dot{\varepsilon}_{ij} - \dot{\varepsilon}_m \delta_{ij}\right) = \sigma_m \dot{\varepsilon}_{ij}\delta_{ij} - \sigma_m \dot{\varepsilon}_{ij}\delta_{ij}\delta_{ij} = \sigma_m \dot{\varepsilon}_{ii} - \sigma_m \dot{\varepsilon}_m \delta_{ii} = \sigma_m 3\dot{\varepsilon}_m - \sigma_m \dot{\varepsilon}_m 3 = 0$$

$$\dot{W}_p = \int_V A_\sigma A_{\dot{\varepsilon}} dV + \int_V D_\sigma D_{\dot{\varepsilon}} dV$$

A – mean hydrostatic tensor
D – deviator

Energy – cont.

For Hooke materials:

$$\dot{W}_p = \int_V A_\sigma A_{\dot{\varepsilon}} dV + \int_V D_\sigma D_{\dot{\varepsilon}} dV$$

$$\left.\begin{matrix} \mathrm{A}_\sigma = 3K\mathrm{A}_\varepsilon & D_\sigma = 2GD_\varepsilon \\ \mathrm{A}_{\dot{\sigma}} = 3K\mathrm{A}_{\dot{\varepsilon}} & D_{\dot{\sigma}} = 2GD_{\dot{\varepsilon}} \end{matrix}\right\} \frac{d}{dt}$$

$$\mathrm{A}_\sigma \mathrm{A}_{\dot{\sigma}} = \frac{1}{2}\frac{d}{dt}(A_\sigma)^2 = \frac{1}{2}2A_\sigma \frac{d}{dt}A_\sigma = A_\sigma A_{\dot{\sigma}}$$

$$\dot{W}_p = \int_V \frac{1}{3K} A_\sigma A_{\dot{\sigma}} dV + \int_V \frac{1}{2G} D_\sigma D_{\dot{\sigma}} dV$$

$$\dot{W}_p = \int_V \frac{1}{6K}\frac{d}{dt}(A_\sigma)^2 dV + \int_V \frac{1}{4G}\frac{d}{dt}(D_\sigma)^2 dV$$

$$W_\mathrm{p} = \int \dot{W}_p dt = \int_V \frac{1}{6K}(A_\sigma)^2 dV + \int_V \frac{1}{4G}(D_\sigma)^2 dV$$

$$\boxed{W_\mathrm{p} = \int_V \Phi_V dV + \int_V \Phi_f dV}$$

$$\Phi_V = \frac{1}{6K}A_\sigma^2 = \frac{1}{2}A_\sigma A_\varepsilon = \frac{3K}{2}A_\varepsilon^2$$

Specific volumetric energy

$$\Phi_f = \frac{1}{4G}D_\sigma^2 = \frac{1}{2}D_\sigma D_\varepsilon = GD_\varepsilon^2$$

Specific distortion energy

Decomposition of the specific energy

$$\Phi = \Phi_v + \Phi_f = \frac{1}{2E}\left[(1+\nu)\sigma_{ij}\sigma_{ij} - \nu\sigma_{kk}^2\right]$$

$$\frac{\partial\Phi}{\partial\sigma_{ij}} = \frac{1}{2E}\left[2(1+\nu)\sigma_{ij} - 2\nu\sigma_{kk}\delta_{ij}\right] = \frac{1}{E}\left[(1+\nu)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij}\right] = \varepsilon_{ij}$$

Specific energy is a potential energy

A general form of specific energy for beams:

$$W_p = \frac{P^2 l}{2AE} = \frac{1}{2}\int_0^l \frac{N^2}{EA}dx$$

$$W_p = \frac{1}{2}\int_0^l \frac{F^2}{S}\mu dx$$

F – cross-sectional force

S – beam stiffness

μ – shape coefficient

Energy – components of the formula

Components of elastic energy formula

Specific case	Cross-sectional force $\boldsymbol{F}$	Beam stiffness $\boldsymbol{S}$	Shape coefficient μ
Tension	N	EA	1
Bending	M	EI	1
Shear	Q	GA	μ
Torsion	M_x	GI_x	μ_t

Generalized forces and displacements

$$\int \frac{dL}{dt} = \frac{dW_p}{dt}\,\mathrm{dt} \qquad\qquad L = W_p$$

External work: function of loading and displacement $=$ Corresponding elastic energy

Definitions of generalised force and generalized displacement:

Generalized force is any external loading in the form of point force, point moment, distributed loading etc.

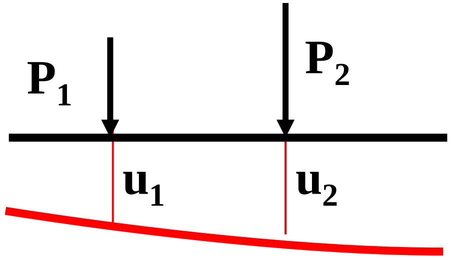
Generalized displacement corresponding to a given generalized force is any displacement for which the work of this force can be performed.

The dimension of generalized displacement has to follow the rules of dimensional analysis taking into account that the dimension of work is [Nm].

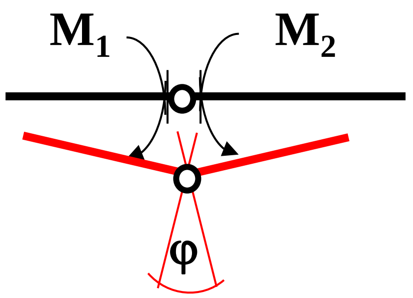
Generalized forces and displacements – cont.

Generalized force	Generalized force dimension	Displacement dimension	Generalized displacement
P	[N]	[m]	u
M	[Nm]	[1]	dw/dx
q	[N/m]	[m^2]	∫udx

But also:



Corresponding generalized displacement is the sum of displacements u_1+u_2



Corresponding generalized displacement is the sum of rotation angles of neighbouring cross-sections φ

Betti principle

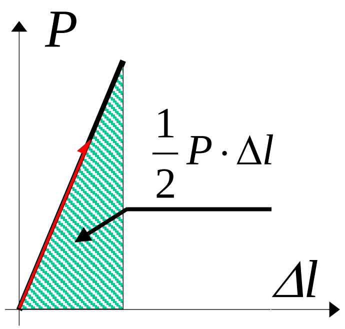
For linear elasticity the principle of superposition obeys:

$$u_i = \sum_j^n P_j \alpha_{ij} \qquad \text{or} \qquad P_i = \sum_j^n u_j \beta_{ij}$$

where α_{ij} β_{ij} are influence coefficients for which Betti principle holds:
$\alpha_{ij} = \alpha_{ji}$ and i $\beta_{ij} = \beta_{ji}$

The work of external forces (generalized) P_i performed on displacements (generalized) u_i is:

$$L = \frac{1}{2}\sum_{i=1}^n P_i u_i = \frac{1}{2}\sum_{i=1}^n \sum_{j=1}^n P_i P_j \alpha_{ij} = \frac{1}{2}\sum_{i=1}^n \sum_{j=1}^n u_i u_j \beta_{ij}$$



After expansion of the first term we have:

$$L = \frac{1}{2}\sum_{i=1}^n P_i u_i = \frac{1}{2}\left(P_1 u_1 + P_2 u_2 + P_3 u_3 + \ldots P_n u_n\right)$$

Castigliano theorem

$$\frac{\partial}{\partial P_1}\left[\quad L=\frac{1}{2}\sum_{i=1}^{n}P_i u_i=\frac{1}{2}\left(P_1u_1+P_2u_2+\ldots P_nu_n\right)\quad\right]$$

taking into account that: $u_i=\sum_j^n P_j\alpha_{ij}=P_1\alpha_{i1}+P_2\alpha_{i2}+\ldots P_n\alpha_{in}$

which after expansion reads:

$$u_1=P_1\alpha_{11}+P_2\alpha_{12}+\ldots+P_n\alpha_{1n}\ ,\quad u_2=P_1\alpha_{21}+P_2\alpha_{22}+\ldots+P_n\alpha_{2n}\ \ldots\quad u_n=P_1\alpha_{n1}+P_2\alpha_{n2}+\ldots+P_n\alpha_{nn}$$

$$\frac{\partial L}{\partial P_1}=\frac{\partial}{\partial P_1}\frac{1}{2}\sum_{i=1}^{n}P_i n_i=\frac{1}{2}\frac{\partial}{\partial P_1}\left(P_1u_1+P_2u_2+P_3u_3+\cdots+P_nu_n\right)=$$

$$=\frac{1}{2}\left(u_1+P_1\frac{\partial u_1}{\partial P_1}+0+P_2\frac{\partial u_2}{\partial P_1}+\ldots+0+P_n\frac{\partial u_n}{\partial P_1}\right)=$$

$$=\frac{1}{2}\left(u_1+\underbrace{P_1\alpha_{11}+P_2\alpha_{21}+\ldots+P_n\alpha_{n1}}_{u_1}\right)=u_1$$

$$\alpha_{21}=\alpha_{12}$$

$$\alpha_{i1}=\alpha_{1i}$$

Unit force theorem

Therefore, for any displacement we have:

$$\frac{\partial L}{\partial P_i} = u_i \qquad \text{and since} \qquad L = W_p \qquad \frac{\partial W_p}{\partial P_i} = u_i$$

To find an arbitrary generalized displacement $\bar{u}$ of any point of the structure one has to apply the corresponding generalized force at this point, and

calculate internal energy associated with all loadings (real and generalized),

take derivative of this energy with respect to generalized force,

and finally set its true value equal to 0:

$$\boxed{\left.\frac{\partial W_p\left(P_i, \bar{P}\right)}{\partial \bar{P}}\right|_{\bar{P}=0} = \bar{u}}$$

$$W_p = \frac{1}{2}\int_0^l \frac{F_i^2}{S}\mu dx$$

Where F_i is cross-sectional force for each case of internal forces reduction (normal force, shear force, bending moment, torsion moment)

Potential energy for bar structures

Making use of superposition principle we have:

$$W_p = W_p\left[F\left(\overline{P}\right) + F\left(P_i\right)\right] \; : \; W_p\left[F\left(\overline{P}=1\right)\cdot \overline{P} + F\left(P_i\right)\right]$$

or

$$W_p = W_p\left[\overline{F}\cdot\overline{P} + F_p\right]$$

$[x]$ denotes here function of x

where:

$$F\left(\overline{P}=1\right) = \overline{F} \qquad F\left(P\right) = F_P$$

With general formula for potential energy:

$$W_p = \frac{1}{2}\int_0^l \frac{F^2}{S}\mu dx$$

we have:

$$W_p = \frac{1}{2}\sum_{j=1}^{n}\sum_{i}^{4}\int_0^l \frac{\left(\overline{F}_i\cdot\overline{P} + F_{Pi}\right)^2}{S_i}\mu_i dx$$

where index *i* has been added
for different reduction cases

Thank you for your attention!