Strength of Materials

9. Stability

Definitions

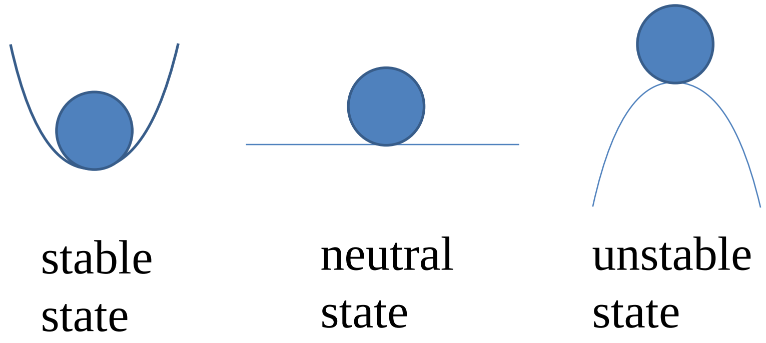
A breaking length – length of a bar that can be self-supported
Steel in tension: with $R = 350$ MPa and $\gamma = 7.8 \cdot 10^4 \text{N/m}^3$, $l = 4.5$ km (the result doesn't depend on the cross-section area)
In compression, the calculation strength of steel is almost the same, but the same result is not acceptable (possible) due to stability loss

The phenomenon of the stability loss can be explained in two ways:

- using potential energy

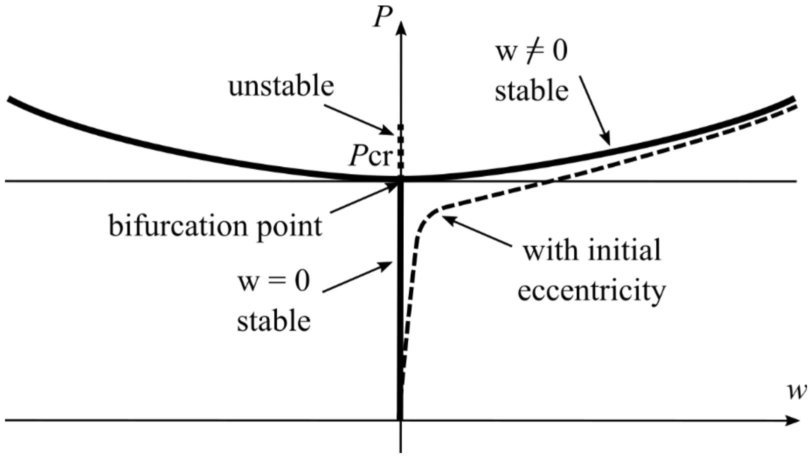
  The system is in a stable state if and only if its potential energy attains minimum.

- using Lapunov's stability theory



a main criterion of a system stability is its response after an infinitesimal stimulus' action:

- in the stable state the system returns to its original position
- in the neutral equilibrium state the system doesn't return to its original position from a neighborhood
- in the unstable state the system buckles significantly and doesn't return to its original position



Definitions – cont.

Why the phenomenon is so dangerous?

- there is no sign of approaching disaster (conf. the diagram on the previous slide), and there is not possibility to make a move
- phenomenon duration is very short (in a split second)
- usually, the loss of stability is catastrophic (important deformations, collapse of whole system and fatalities)

STABILITY – a state of permanent equilibrium of a structural member
BUCKLING – a law of a structure movement during or after stability loss

Deformations during the stability loss:

- before – only axial (the state of simple compression)
- during – undetermined (but bending deflections are observed)
- after – a new state of stability can be observed with important deformations

For a solution in the deformed state, the principle of solidification cannot be used any more, due to important deformations of the system. It means, that there always is nonlinear geometry.
In the actual (deformed) configuration, bending moments arise, resulting from eccentric action of axial force and from boundary conditions:

$$M(x) = Pw(x) + M_0(x)$$

Solution in the linear elastic state

In the linear elastic state

$$EIw''(x) = -M(x)$$

$$EIw''(x) + Pw(x) = M_0(x)$$

$$w''(x) + \frac{P}{EI}w(x) = \frac{M_0(x)}{EI}$$

$$k^2 \stackrel{\text{def}}{=} \frac{P}{EI}$$

$$\boxed{w''(x) + k^2 w(x) = -\frac{M_0(x)}{EI}}$$

This is an Euler's equation.
The complementary function is:

$$w_{\text{comp}}(x) = A\sin kx + B\cos kx$$

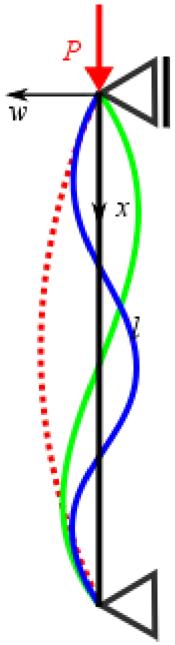
The particular integral depends on the form of the right hand side of the Euler's equation.
The integral constants as well as the right hand side of the Euler's equation should be determined taking into account the boundary conditions:

$$w(x) = A\sin kx + B\cos kx + w_{\text{part}}(x)$$

Solution to a pin-ended strut

$$M_0(x) = 0 \rightarrow w_{\text{part}}(x) = 0$$

$$w(x) = A \sin kx + B \cos kx$$



the boundary conditions:

$n = 1$

$n = 2$

$n = 3$

$$w(0) = 0 \rightarrow 0 = A \cdot 0 + B \cdot 1 \rightarrow B = 0$$

$$w(l) = 0 \rightarrow 0 = A \sin kl \rightarrow A = 0 \quad \text{or} \quad \sin kl = 0$$

$A = 0 \rightarrow w(x) \equiv 0$ not acceptable trivial solution

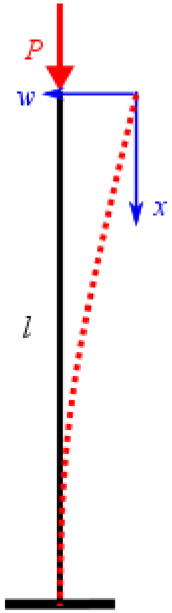
$$\sin kl = 0 \rightarrow kl = n\pi \rightarrow k^2 = \frac{n^2\pi^2}{l^2} \rightarrow P_{cr} = \frac{n^2\pi^2 EI}{l^2}$$

minimum value is approached for $n = 1$ and $I = I_{\text{min}}$

$$\boxed{P_{cr} = \frac{\pi^2 EI_{\text{min}}}{l^2}}$$

Solution to a pole

$$M_0(x) = 0 \rightarrow w_{\text{part}}(x) = 0$$



$$w(x) = A\sin kx + B\cos kx$$

the boundary conditions:

$$w(0) = 0 \rightarrow 0 = A\cdot 0 + B\cdot 1 \rightarrow B = 0$$

$$w'(l) = 0 \rightarrow 0 = Ak\cos kl = 0 \rightarrow A = 0 \quad \text{or} \quad \cos kl = 0$$

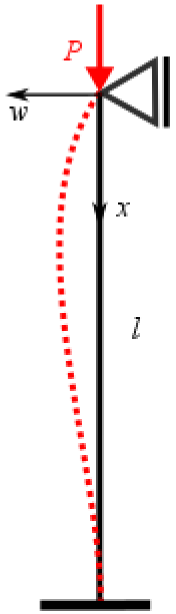
$A = 0 \rightarrow w(x) \equiv 0$ not acceptable trivial solution

$$\cos kl = 0 \rightarrow kl = \frac{\pi}{2} + n\pi \rightarrow kl = \frac{\pi}{2} \rightarrow k^2 = \frac{\pi^2}{4l^2} \rightarrow P_{cr} = \frac{\pi^2 EI}{4l^2}$$

minimum value is approached for $I = I_{\min}$

$$\boxed{P_{cr} = \frac{\pi^2 EI_{\min}}{4l^2}}$$

Solution to a supported pole



$$M_0(x) = Rx \rightarrow w_{\text{part}}(x) = -\frac{R}{P}x$$

$$w(x) = A\sin kx + B\cos kx - \frac{R}{P}x$$

the boundary conditions:

$$w(0) = 0 \rightarrow 0 = A \cdot 0 + B \cdot 1 - 0 \rightarrow B = 0$$

$$w'(l) = 0 \rightarrow 0 = Ak\cos kl - \frac{R}{P} = 0 \rightarrow R = APEIk\cos kl$$

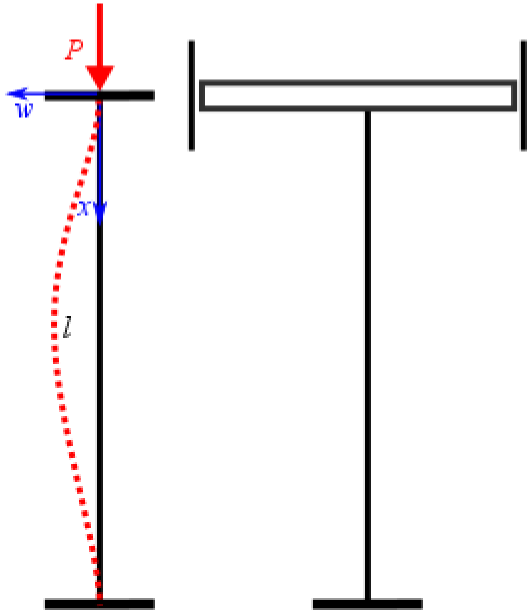
$$w(l) = 0 \rightarrow 0 = A\sin kl + \frac{APk\cos kl}{P}l \rightarrow \quad \tan kl = kl \rightarrow kl = 4.493$$

$$P = \frac{20.19\pi^2 EI}{l^2} = \frac{\pi^2 EI}{(0.699l)^2}$$

minimum value is approached for $I = I_{\text{min}}$

$$\boxed{P_{cr} = \frac{\pi^2 EI_{\text{min}}}{(0.699l)^2}}$$

Solution to the bilaterally fixed beam



$$M_0(x) = Rx - M_1 \rightarrow w_{\text{part}}(x) = -\frac{R}{P}x + \frac{M_1}{P}$$

$$w(x) = A\sin kx + B\cos kx - \frac{R}{P}x + \frac{M_1}{P}$$

the boundary conditions:

$$w(0) = 0 \rightarrow 0 = A\cdot 0 + B\cdot 1 - 0 + \frac{M_1}{P} \rightarrow M_1 = -BP$$

$$w'(0) = 0 \rightarrow 0 = Ak\cos 0 - Bk\sin 0 - \frac{R}{P} = 0 \rightarrow R = APk$$

$$w'(l) = 0 \rightarrow 0 = Ak\cos kl - Bk\sin kl - Ak \rightarrow \quad B = A\frac{(\cos kl - 1)}{\sin kl}$$

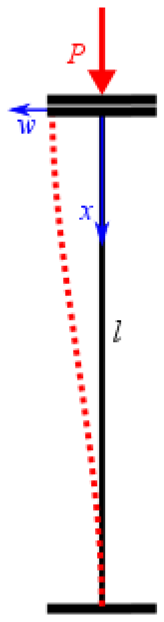
$$w(l) = 0 \rightarrow A\sin kl + B\cos kl - \frac{R}{P}l + \frac{M_1}{P} = 0 \rightarrow 1 - \cos kl = kl\sin kl$$

$$kl = 2\pi \rightarrow P_{cr} = \frac{4\pi^2 EI}{l^2}$$

minimum value is approached for $I = I_{\min}$

$$\boxed{P_{cr} = \frac{\pi^2 EI_{\min}}{(0.5l)^2}}$$

Solution to a pole with guided support



$$M_0(x) = M_1 \rightarrow w_{\text{part}}(x) = -\frac{M_1}{P}$$

$$w(x) = A\sin kx + B\cos kx - \frac{M_1}{P}$$

the boundary conditions:

$$w'(0) = 0 \rightarrow 0 = Ak\cos 0 - Bk\sin 0 = 0 \rightarrow A = 0$$

$$w(l) = 0 \rightarrow 0 = A\sin kl + B\cos kl - \frac{M_1}{P} \rightarrow M_1 = BP\cos kl$$

$$w'(l) = 0 \rightarrow 0 = Ak\cos kl - Bk\sin kl \rightarrow \sin kl = 0 \rightarrow kl = \pi$$

$$P_{cr} = \frac{\pi^2 EI}{l^2}$$

minimum value is approached for $I = I_{\min}$

$$\boxed{P_{cr} = \frac{\pi^2 EI_{\min}}{l^2}}$$

General formula for Euler's critical force

the effective length:

$$l_e \overset{\text{def}}{=} \alpha l$$

where l is the actual bar length and the parameter α depends on the static scheme

- $\alpha = 1$ for a pin-ended element
- $\alpha = 0.5$ for a pole
- $\alpha = 0.699$ for a supported pole
- $\alpha = 0.5$ for a bilaterally fixed beam
- $\alpha = 1$ for a pole with guided support

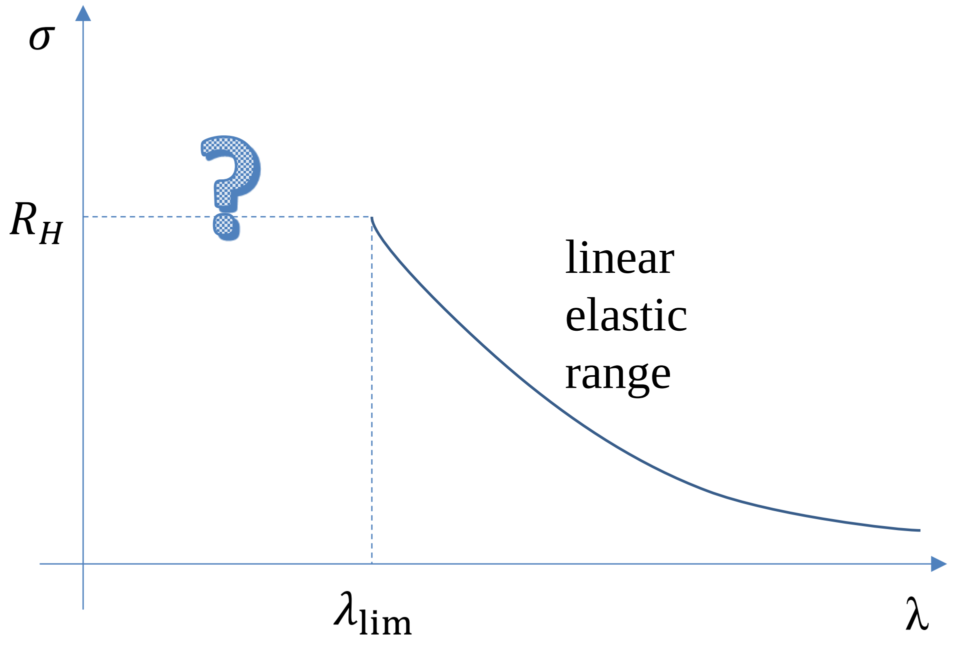
the slenderness ratio:

$$\lambda \overset{\text{def}}{=} \frac{l_e}{i_{min}}$$

The Euler's formula for the (Euler's) critical force in a general form:

$$P_E = \frac{\pi^2 E I_{\min}}{l_e^2} = \frac{\pi^2 E A}{(\lambda_{\max})^2}$$

Validity range for the Euler's formula



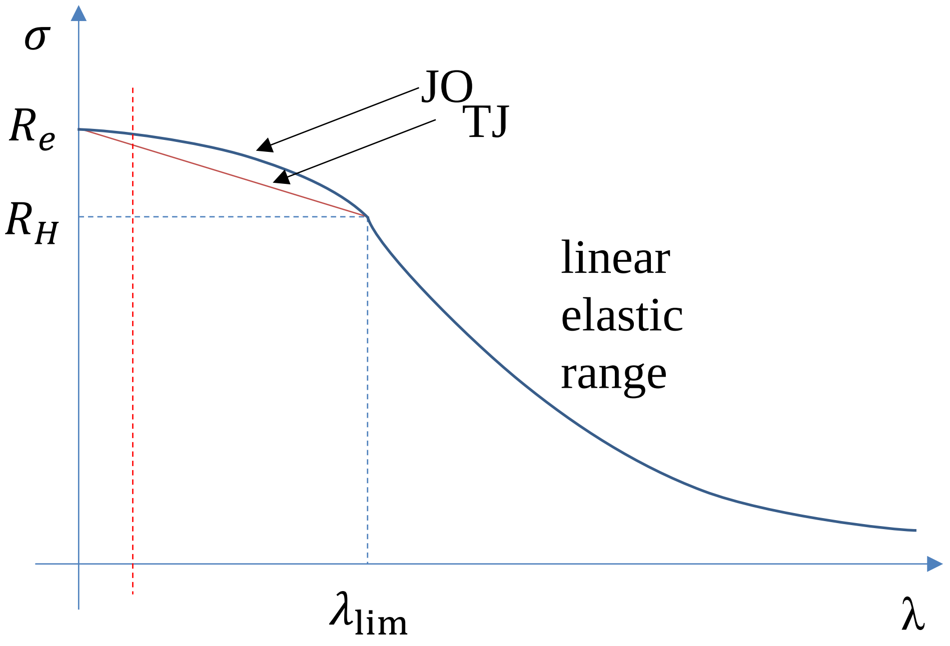
The Euler's formula is valid for linearly elastic material

it means, that $\sigma \leq R_H \rightarrow \frac{P_E}{A} \leq R_H \rightarrow \frac{\pi^2 E}{\lambda_{max}^2} \leq R_H$, so:

$$\lambda_{lim} \geq \pi \sqrt{\frac{E}{R_H}}$$

steel: $\lambda_{lim} \approx 65 \div 90$
aluminum: $\lambda_{lim} \approx 46 \div 64$
oak: $\lambda_{lim} \approx 55$
concrete: $\lambda_{lim} \approx 85$

Solution beyond the linear elastic range



$$P_{T-J} = A\sigma_{T-J}$$

$$P_{J-O} = A\sigma_{J-O}$$

Tetmeyer-Jasiński formula:

$$\boxed{\sigma_{T-J} = a - b\lambda}$$

where:

$$\sigma_{T-J}(0) = R_e \rightarrow \; a = R_e,$$

$$\sigma_{T-J}(\lambda_{\min}) = R_H \rightarrow b = \frac{R_e - R_H}{\lambda_{\min}}$$

Johnson-Ostenfeld formula:

$$\boxed{\sigma_{J-O} = A - B\lambda^2}$$

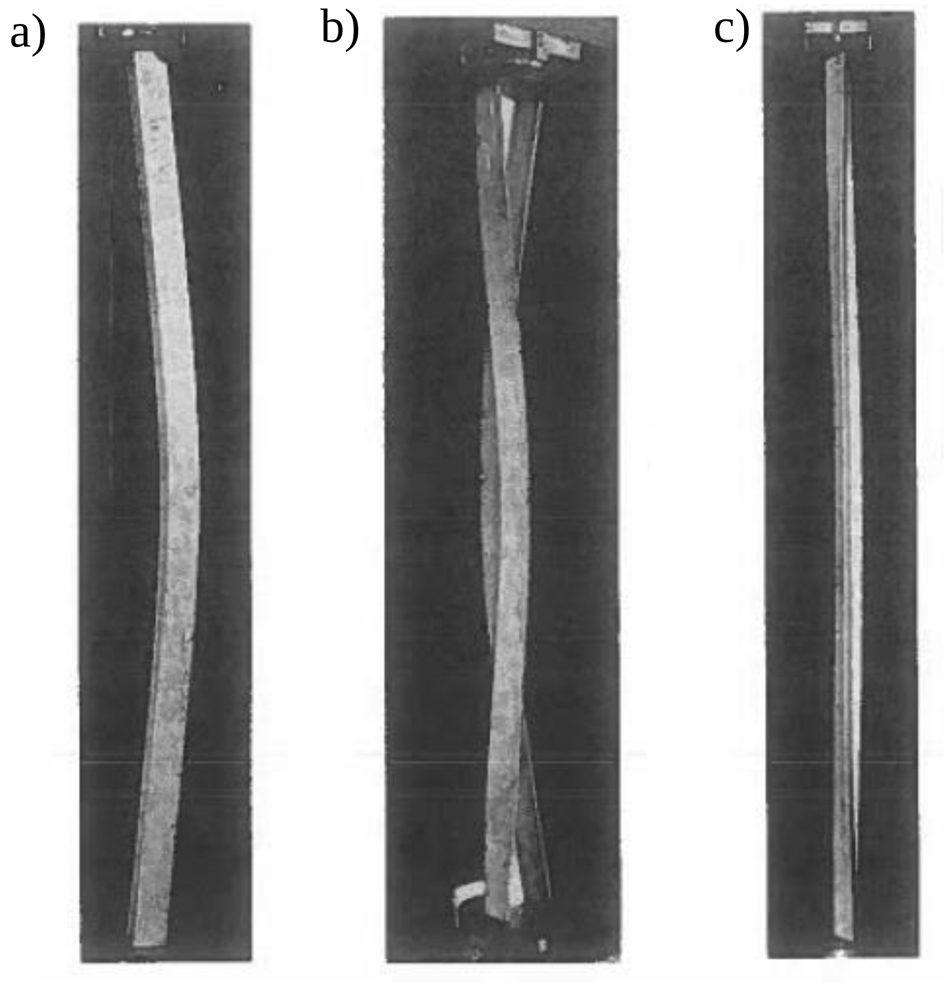
where:

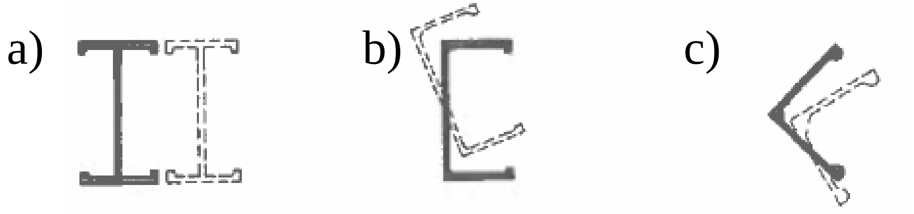
$$\sigma_{J-O}(0) = R_e \rightarrow C = R_e$$

$$\sigma_{J-O}(\lambda_{\lim}) = \mathrm{R_H} \rightarrow \mathrm{B} = \frac{\mathrm{R_e} - \mathrm{R_H}}{\lambda_{\lim}^2}$$

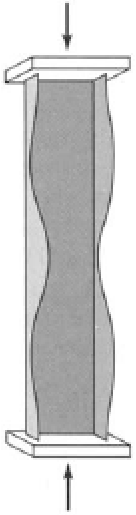
if $\lambda < 10$, the danger of the stability loss may be neglected

Different buckling phenomena





a) plane buckling or buckling by bending
b) torsional buckling
c) spatial buckling or buckling by bending and torsion



local buckling

Dimensioning

Keep in mind, that the accident risk is always present in compression!

Because of catastrophic consequences of the stability loss, the safety coefficients are really huge: from 1.5 for small values of the slenderness ratio, to even 7 for great values of the slenderness ratio.

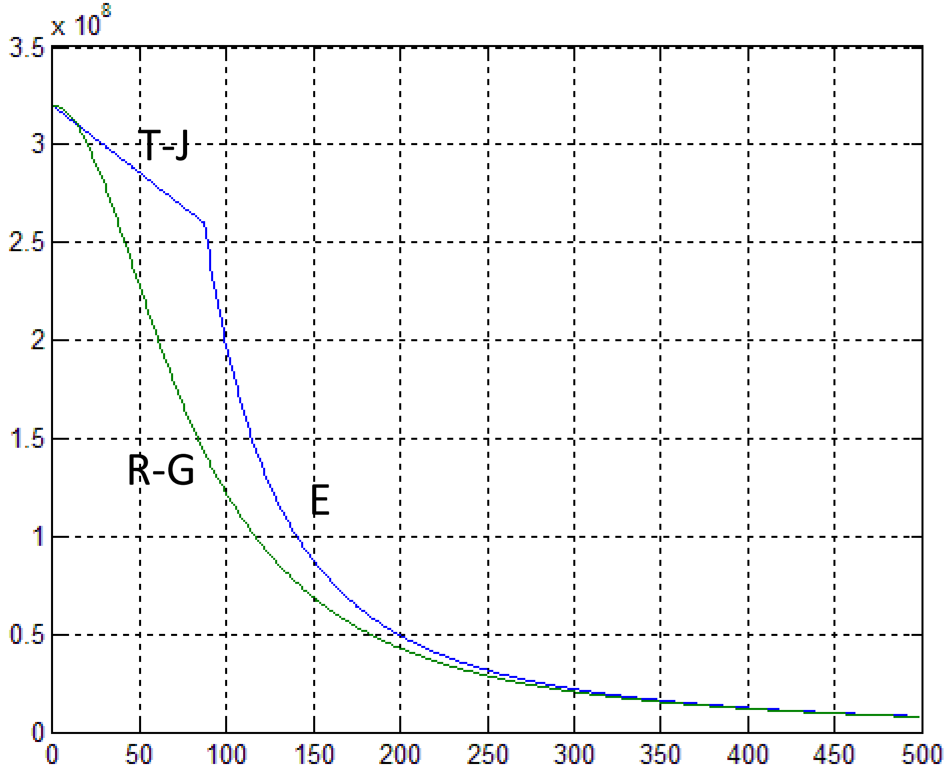
Despite great safety coefficients, the slenderness ratio is usually limited from 10 to 350 about.

$$P_{acc} = \frac{P_E, P_{T-J}, P_{J-O}}{n}$$

$$n = 1.5 \div 7$$

!

Rankine-Gordon formula



Rankine-Gordon formula:

$$\frac{1}{P_{R-G}} = \frac{1}{P_E} + \frac{1}{AR_e}$$

$$\sigma_{R-G} = \frac{\pi^2 E R_e}{R_e \lambda^2 + \pi^2 E}$$

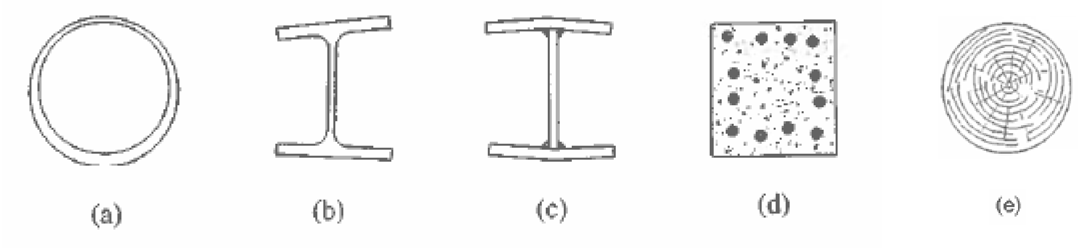
$$P_{R-G} = \frac{AR_e}{1 + \left(\frac{\lambda}{\lambda_c}\right)^2}, \qquad \lambda_c = \pi\sqrt{\frac{E}{R_e}}$$

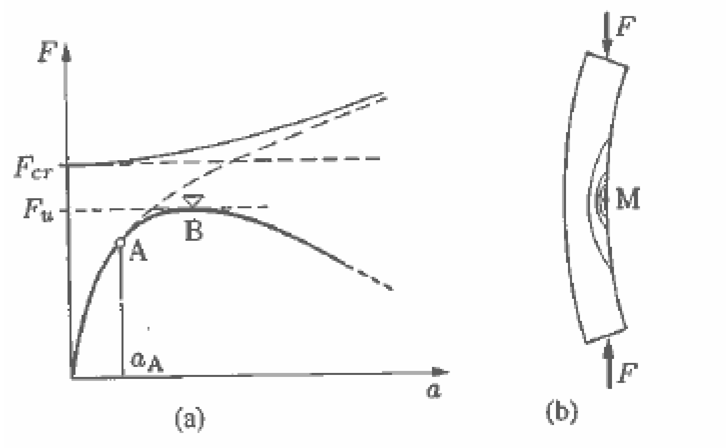
Material	λ_c	R_e[Mpa]
mild steel	87	300
wrought iron	89	250
cast iron	135	560
timber	31	35

Effect of imperfections

Imperfections:

a) tube eccentricity
b) rolling tolerance
c) welding distortions
d) rebars shifted positions
e) material parameters variations

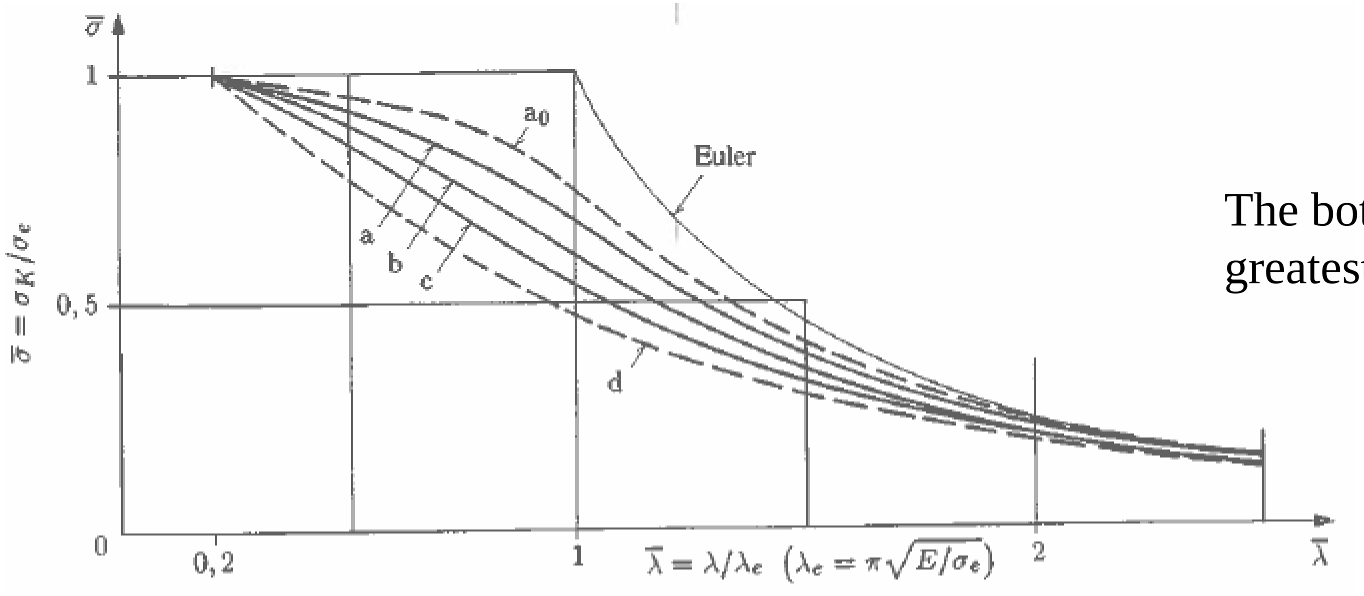




Real member behavior:

a) buckling
b) plastic zone localization

Steel buckling curves



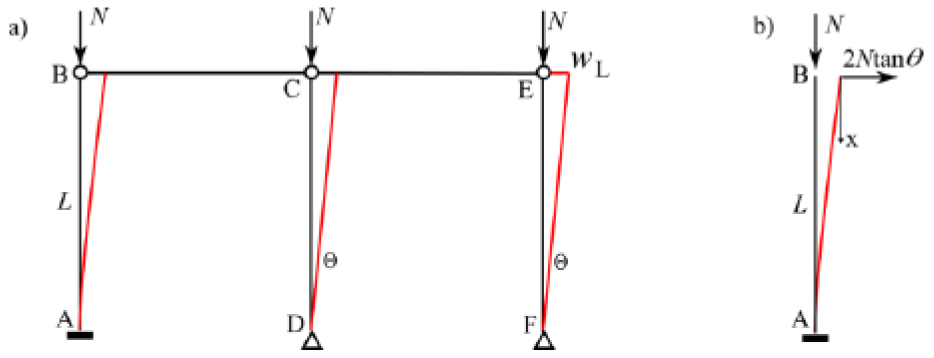
The bottom curve refers to the members with the greatest imperfections

The buckling curves of industrial steel members

General rule of thumb: the higher slenderness , the greater bearing capacity loss (and the greater safety coefficients should be adopted)

Stability of a system – an example

Determine the critical load of the structure ABCDEF in the Fig. below, composed of one fixed-end column and several bi-articulated beams and columns. Compare the effective length of the fixed column with the regular value of the basic case. Determine the critical load in the case of one-bay ABCD frame.



Solution

in actual configuration (Fig. b):

$$M(x) = Nw + 2Nx\tan\theta$$

using $EIw''(x) = -M(x)$ and $k^2 \overset{\text{def}}{=} \frac{N}{EI}$

we get the Euler's equation:

$$w'' + k^2 w = -2k^2 x \tan\theta$$

with the complementary and particular functions:

$w(x) = A\sin kx + B\cos kx - 2x\tan\theta$

KBC: $w(0) = 0; w'(L) = 0 \rightarrow B = 0, Ak\cos kL - 2\tan\theta = 0$

and with $w(L) = w_L \rightarrow A\sin kL - 2L\tan\theta = L\tan\theta \rightarrow \ \tan\theta = \frac{A}{3L}\sin kL$, we get:

$$kL\cos kL - \frac{2}{3}\sin kL = 0 \rightarrow kL = 0.9674 \rightarrow (kL)^2 = 0.9359 \rightarrow N_{cr} = \frac{0.9359EI}{L^2} \rightarrow l_e = 3.247L \gg 2L$$

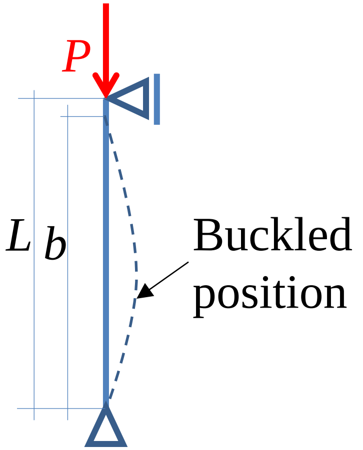
For the one bay frame, using the same method, the result is $N_{cr} = \frac{1.3586EI}{L^2}, \rightarrow l_e = 2.695L \gg 2L$

Energy method

The energy method is based on the first law of thermodynamics. The increase in internal energy is equal to the change in strain energy. For an adiabatic elastic column:

$$\delta W = \delta U$$

The column may buckle when the load first reaches a value for which the above equality is fulfilled.



external work variation

$$ds^2 = dx^2 + dz^2 \rightarrow ds = \sqrt{1 + (w')\text{^}2}\, dx = \left(1 + \frac{1}{2}(w')^2 + \cdots\right) dx$$

$$L = \int_0^b \left(1 + \frac{1}{2}(w')^2\right) dx = b + \frac{1}{2}\int_0^b (w')^2 dx \cong b + \frac{1}{2}\int_0^L (w')^2 dx$$

$$\delta W = P(L - b) = \frac{P}{2}\int_0^L (w')^2\, dx$$

internal energy variation

$$\delta U = U_1 - U_0 = U_1 = \int_0^L \frac{M^2}{2EI} dx = \frac{1}{2}\int_0^L EI(w'')^2\, dx$$

$$P_{cr} = \frac{\int_0^L EI(w'')^2 dx}{\int_0^L (w')^2 dx}$$

Energy method – cont.

Let us assume $w(x)$ in the Fourier series:

$$w(x) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L}$$

then we get

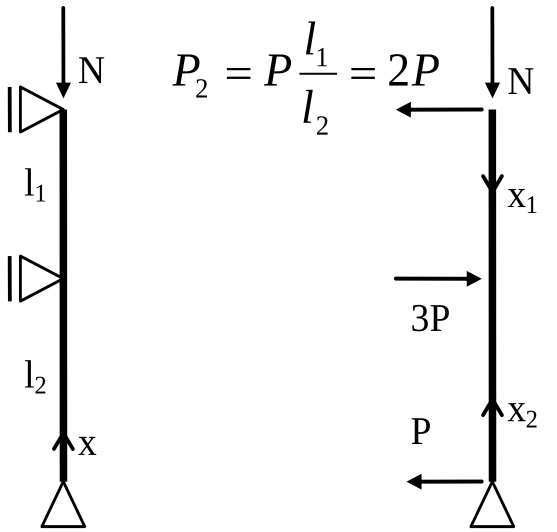
$$P_{cr} = \frac{\pi^2 EI}{L^2} \frac{\sum_{n=1}^{\infty} n^4 a_n^2}{\sum_{n=1}^{\infty} n^2 a_n^2}$$

for:

- $a_1 \neq 0$, other $a_i = 0 \rightarrow \quad P = \frac{\pi^2 EI}{L^2}$
- $a_2 \neq 0$, other $a_i = 0 \rightarrow \quad P = \frac{4\pi^2 EI}{L^2} = \frac{\pi^2 EI}{(0.5L)^2}$
- (and so on…)

Energy method – an example

Determine the Euler's critical force for a two-story prismatic column in the Fig. below; $l_1 = 4$ m, $l_2 = 2$ m.



Solution

The first approximation: $w(x) = a(x^3 + bx^2 + cx + d)$

KBC: $w(0) = w(l_1) = w(l_1 + l_2) = 0$

$\mathrm{w}(x) = a(x^3 - 10x^2 + 24x)$

$N_{cr} = 1.458\ EI$

The 2nd approximation: $w(x) = a(x^5 + bx^4 + cx^3 + dx^2 + ex + f)$

KBC: $w(0) = w(l_1) = w(l_1 + l_2) = 0$

SBC: $w''(0) = w''(l_1 + l_2) = 0$

$$w(x) = \cdots$$

$$N_{cr} = 0.952\ EI$$

The exact solution

Using the complementary and particular functions for both equations *1* and *2* we get:

$w_1(x_1) = A\sin k_1 x_1 + B\cos k_1 x_1 - \frac{P}{N}x_1,\ \ w_2(x_2) = C\sin k_2 x_2 + D\cos k_2 x_2 - \frac{P}{N}x_2$

KBC: $w_1(0) = w_1(l_1) = 0, w_2(0) = w_2(l_2) = 0$, the compat. conditions: $w_1'(l_1) = -w_2'(l_2), w_1''(l_1) = w_2''(l_2)$

and finally we get $k\left(\frac{\cos kl_1}{\sin kl_1} + \frac{\cos kl_2}{\sin kl_2}\right) = \frac{l_1 + l_2}{l_1 l_2} \rightarrow k = 0.9642 \rightarrow N_{cr} = k^2 EJ = 0.930EJ$

That’s all, folks!