# Strength of Materials

## 11. Plasticity

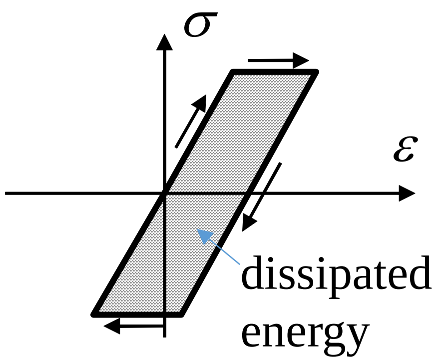
# Definitions

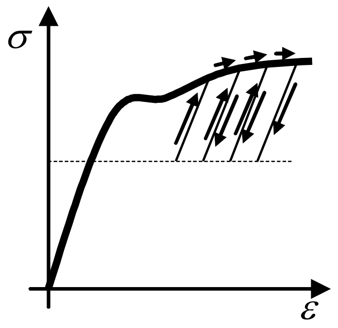
Elastic materials when unloaded return to initial state (the strains caused by load are reversible)

Plastic strains occur when loads are high enough

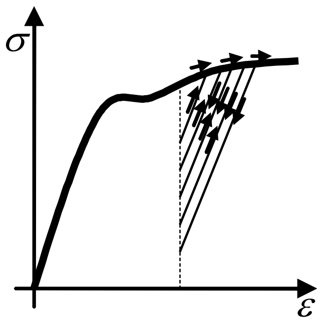
Plasticity – a permanent deformation of a solid material.
In the most cases unloading is an elastic process.
Plastic deformations are irreversible.
There is no cycle in plasticity.
Instead, there are quasi-cycles:

- stress quasi-cycle
- strain quasi-cycle

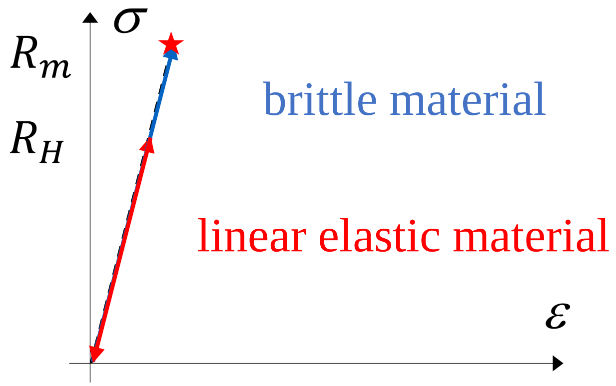


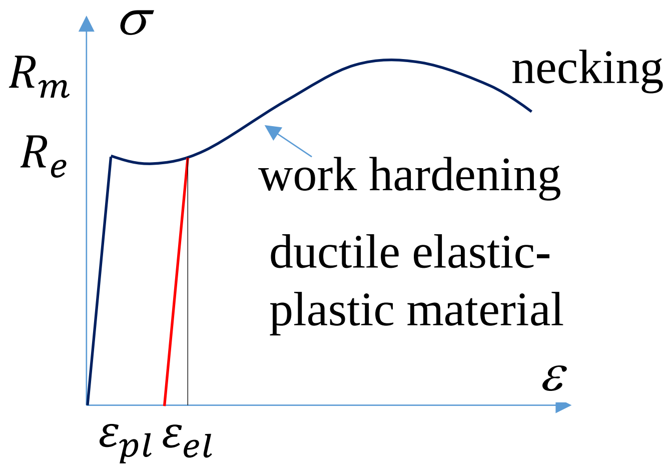




stress quasi-cycle



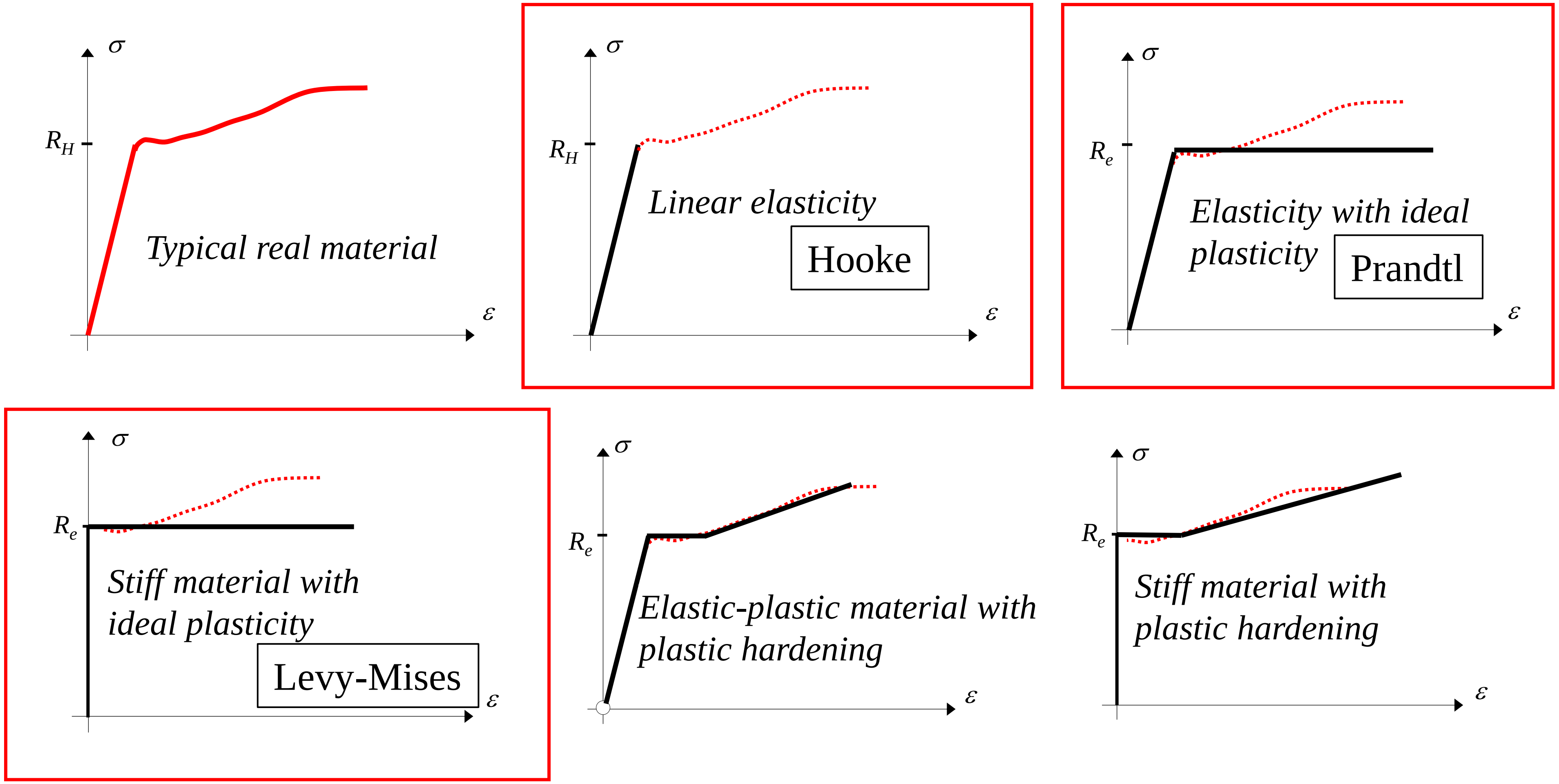

strain quasi-cycle



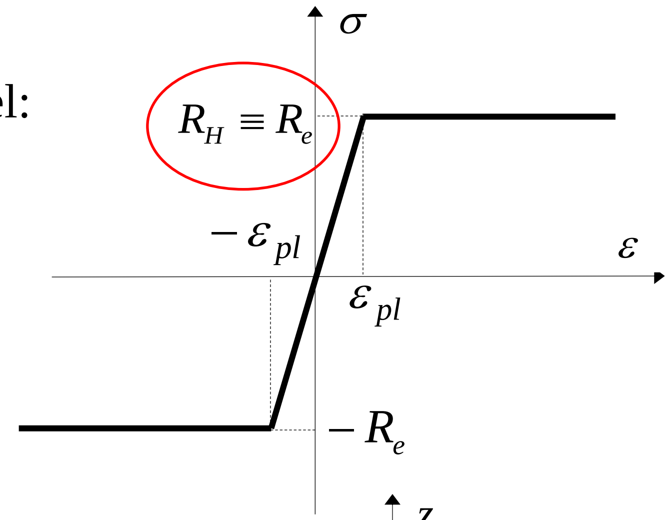
# Schematization

Different idealizations of tensile diagram for elastic-plastic materials



# Elastic-plastic bending – elastic range

Prandtl model:




$-\varepsilon_{pl} \leq \varepsilon \leq \varepsilon_{pl} \quad \rightarrow \quad \sigma = E\varepsilon$

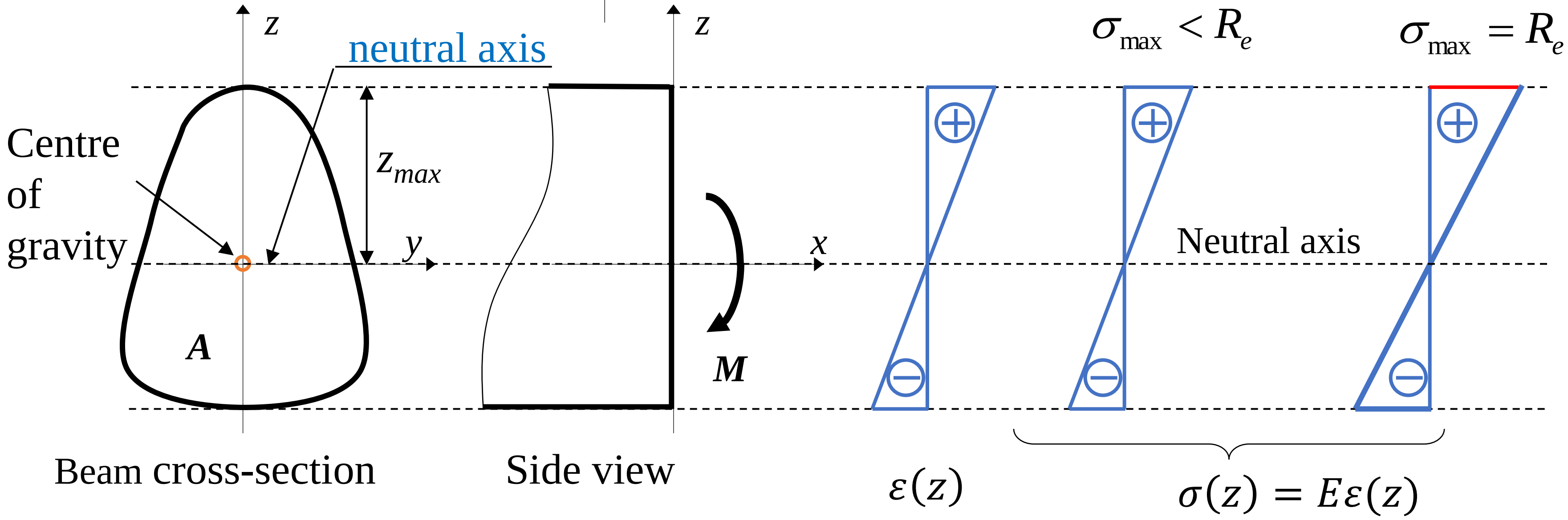
$\varepsilon > \varepsilon_{pl} \quad \rightarrow \quad \sigma = R_e$

$\varepsilon < -\varepsilon_{pl} \quad \rightarrow \quad \sigma = -R_e$

## Elastic range

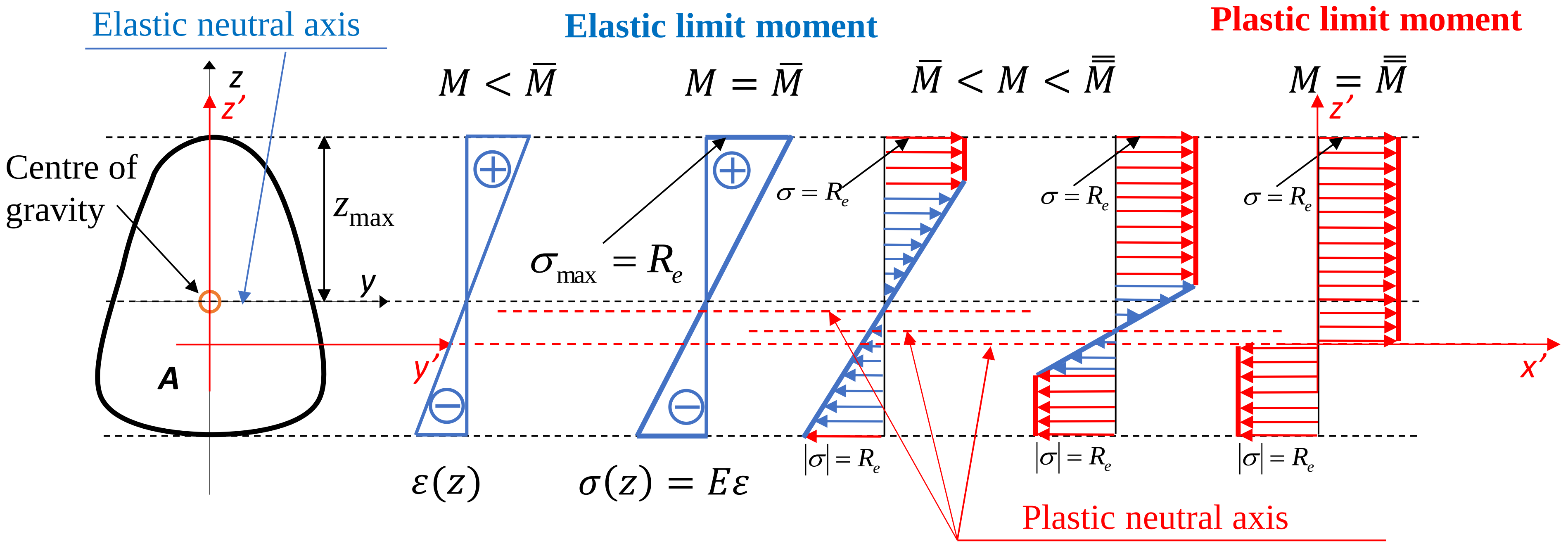
$M < \overline{M}$ $\quad$ $M = \overline{M}$

$\sigma_{\max} < R_e$ $\quad$ $\sigma_{\max} = R_e$




symbols:
$\overline{M} \equiv M_{el}$

# Elastic-plastic bending – plastic range




symbols: $\bar{M} \equiv M_{el}$, $\bar{\bar{M}} \equiv M_{pl}$

# Parameters describing stress distribution

Assumptions of so-called technical bending theory:

- uniaxial stress state
- plane cross-sections hypothesis

Bernoulli's hypothesis: $\varepsilon = \varepsilon_0 + \kappa z$

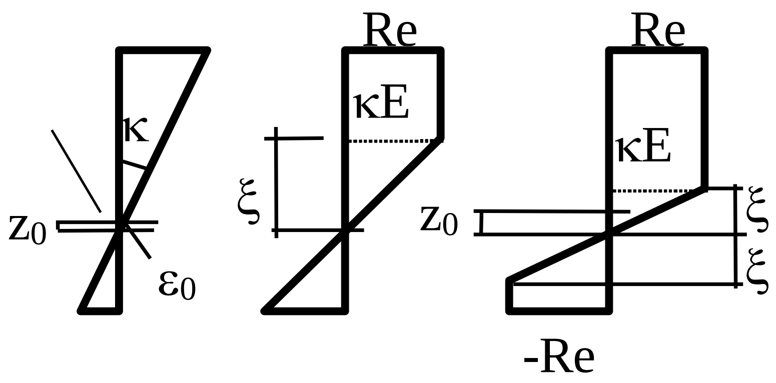
Parameters that describes stress distribution:

- neutral axis position, $z_0$
- plastic front position, $z_p$
- axis curvature, κ
- axis line strain, $\varepsilon_0$
- elastic range extent, ξ

(but only two independent)

material parameters:

- Young modulus, $E$
- yield stress, $R_e$




position of neutral axis:

$$\varepsilon(z_0) = 0 \ \rightarrow \ z_0 = -\frac{\varepsilon_0}{\kappa}$$

plastic front position:

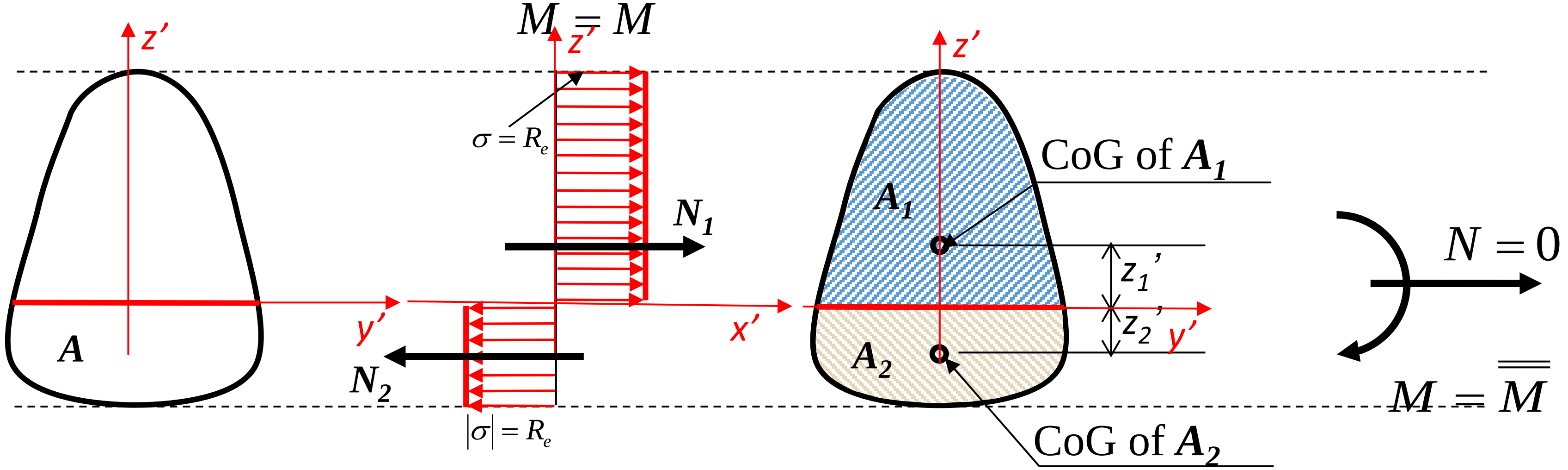
$$\varepsilon_0 + \kappa z_p = \pm\frac{R_e}{E} \ \rightarrow z_p = \frac{1}{\kappa}\left(\pm\frac{R_e}{E} - \varepsilon_0\right)$$

elastic range extent:

$$E\kappa\xi = R_e \ \rightarrow \ \xi = \frac{R_e}{E\kappa}$$

# Elastic-plastic bending

**Plastic limit moment**




$$N = 0 = \int_A \sigma(z')\,dA = N_1 - N_2 = A_1 \cdot R_e - A_2 \cdot R_e \quad \Longrightarrow \quad A_1 = A_2 = \frac{A}{2}$$

$$\overline{\overline{M}} = \int_A \sigma(z')z'\,dA = N_1 \cdot z'_1 + N_2 \cdot z'_2 = R_e A_1 \cdot z'_1 + R_e A_2 \cdot z'_2 = R_e\left(S'_1 + S'_2\right)$$

$$\overline{\overline{M}} = R_e\left(S'_1 + \left|S'_2\right|\right) = R_e\left(\frac{A}{2}z'_1 + \frac{A}{2}\left|z'_2\right|\right) = AR_e \cdot \left(z'_1 + \left|z'_2\right|\right)/2$$

$$\overline{\overline{M}} = R_e \cdot \left(\left|z'_1\right| + \left|z'_2\right|\right)(A/2) = R_e W_{pl}$$

# Elastic-plastic bending – shape coefficient

**Limit elastic moment**

$M = \bar{M} \qquad \sigma_{\max} = \bar{M}/W_{el} = \mathrm{R_e}$

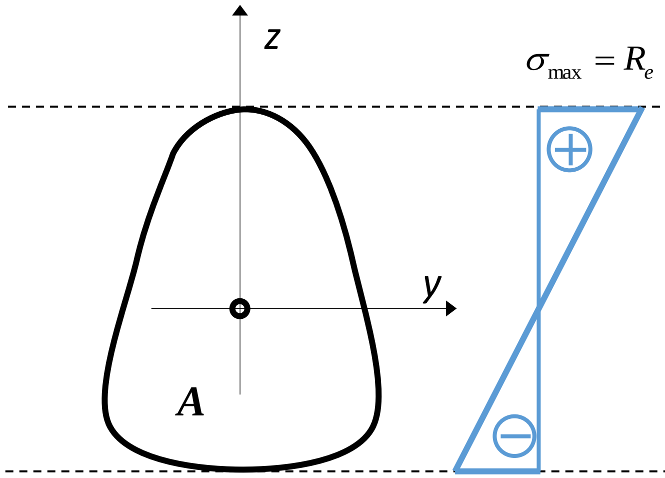
$$\bar{M} = R_e W_{el} \qquad W_{el} = \bar{W} = I_y/z_{\max}$$

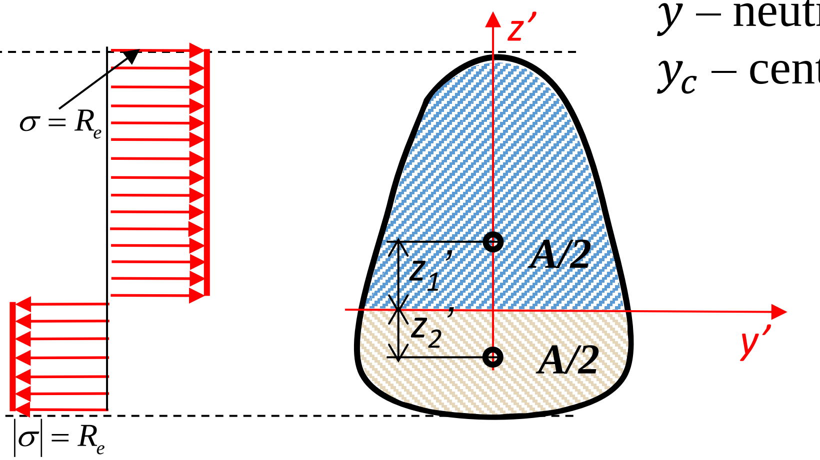
**Limit plastic moment**

$M = \bar{\bar{M}}$

$$\bar{\bar{M}} = R_e W_{pl} \qquad W_{pl} = \bar{\bar{W}} = (|z_1'| + |z_2'|) \cdot \frac{A}{2}$$

$$W_{pl} = \left|S_y^{(1)}\right| + \left|S_y^{(2)}\right| = 2\left|S_{y_c}^{(1)}\right| = 2\left|S_{y_c}^{(2)}\right|$$



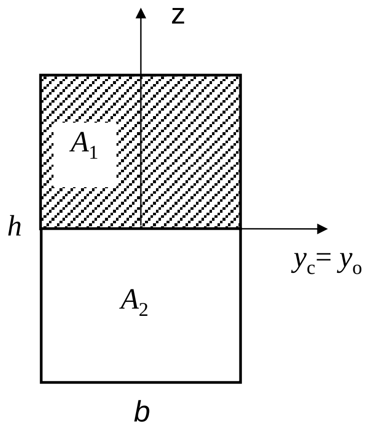

$y$ – neutral or any axis

$y_c$ – central axis

$$\frac{\bar{\bar{M}}}{\bar{M}} = \frac{W_{pl}}{W_{el}} = k$$

$k \geq 1$ – shape coefficient

# Shape coefficient – cont.



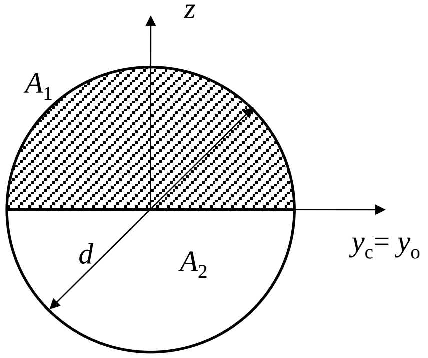

$$W_{el} = \frac{bh^2}{6}$$

$$M_{el} = R_e \frac{bh^2}{6}$$

$$W_{pl} = 2S_{y_c}(A_1) = 2b \cdot \frac{h}{2} \cdot \frac{h}{4} = \frac{bh^2}{4}$$

$$W_{pl} = S_{y_0}(A_1) - S_{y_0}(A_2) = b \cdot \frac{h}{2} \cdot \frac{h}{4} - \left[ b \cdot \frac{h}{2} \left( -\frac{h}{4} \right) \right] = \frac{bh^2}{4}$$

$$\bar{\bar{M}} = R_e \frac{bh^2}{4} \qquad k = \frac{M_{pl}}{M_{el}} = \frac{W_{pl}}{W_{el}} = 1.5$$

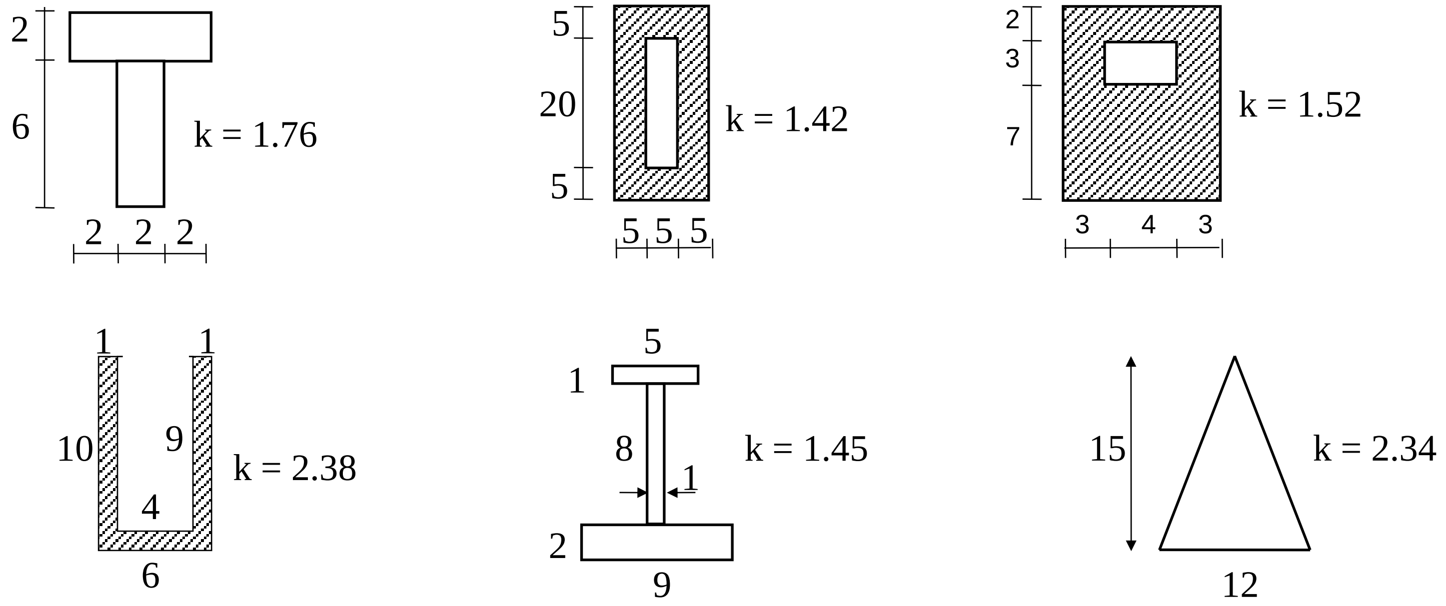



$$W_{el} = \frac{\pi d^3}{32}$$

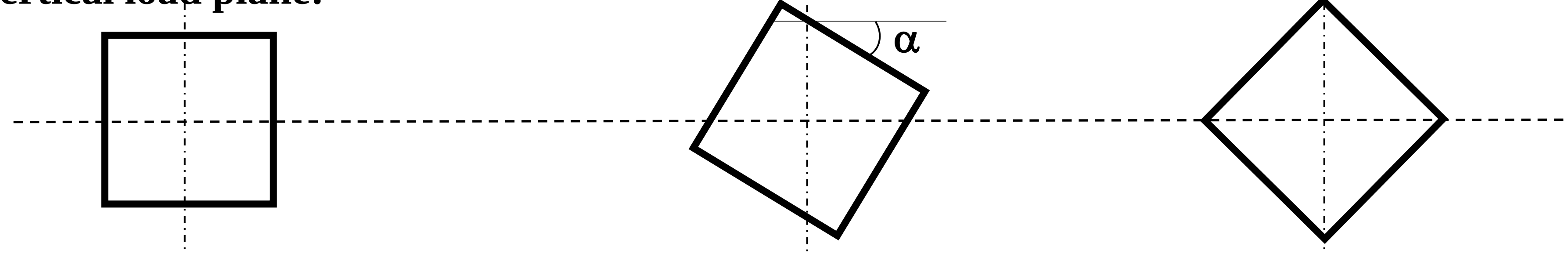
$$W_{pl} = 2S_{y_c}(A_1) = 2 \cdot \frac{1}{2} \cdot \frac{\pi d^2}{2} \cdot \frac{4}{3\pi} \cdot \frac{d}{2} = \frac{d^3}{6}$$

$$k = \frac{M_{pl}}{M_{el}} = \frac{W_{pl}}{W_{el}} = 1.7$$

# Shape coefficient – cont.

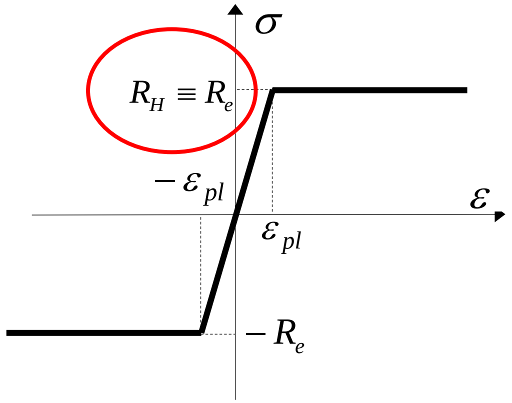



**vertical load plane:**




$k = 1.5$ $\quad$ $k = k(\alpha) =?$ $\quad$ $k = 2$

# Cross-section elastic and plastic limits



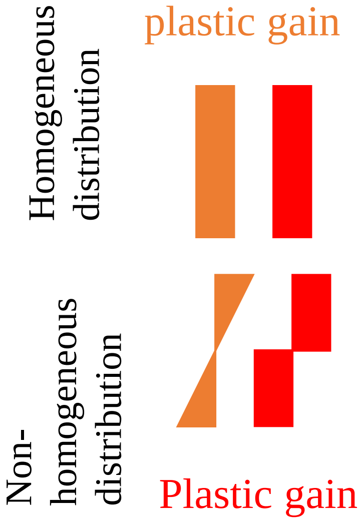

**Elasticity with ideal plasticity**

**Statically determined**

| | Limit elastic capacity | Limit plastic capacity | Ratio of pl/el capacities $k$ |
|---|---|---|---|
| tension | $\bar{N} = R_e A$ | $\bar{\bar{N}} = R_e A$ | 1 |
| bending | $\bar{M} = R_e W_{el}$ | $\bar{\bar{M}} = R_e W_{pl}$ | $k \geq 1$ |

**Statically undetermined**

No plastic gain

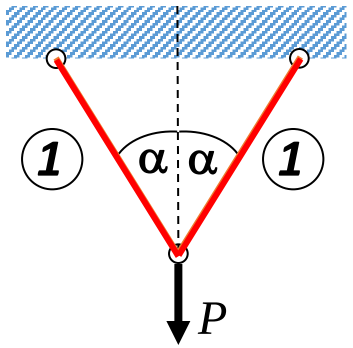



Plastic gain

# Limit analysis of structures

**Statically determined structures**

Length and cross-section area of both bars: ***l, A***




$\overline{P} = \overline{\overline{P}}$

**Elastic solution**

**Plastic solution**

From equilibrium:

$$N_1 = \frac{P}{2\cos\alpha}$$

Stress in bars:

$$\sigma_1 = \frac{P}{2A\cos\alpha}$$

In limit elastic state:

$$R_e = \frac{\overline{P}}{2A\cos\alpha}$$

Limit elastic capacity:

$$\overline{P} = 2AR_e\cos\alpha$$

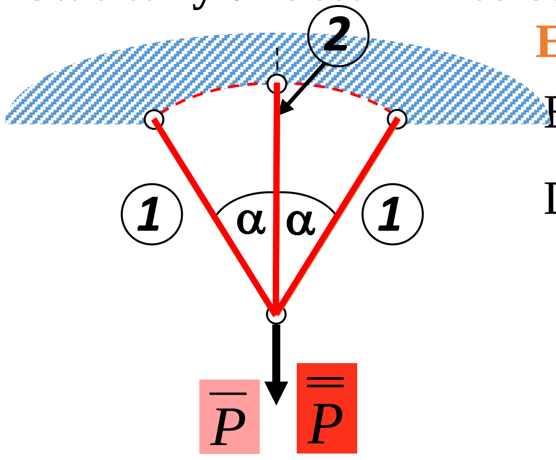
Limit plastic capacity:

$$\overline{\overline{P}} = 2AR_e\cos\alpha$$

# Limit analysis of structures – cont.

**Statically undetermined structures**

Length and cross-section area of the bars: ***l, A***




**Elastic solution**

Equilibrium : $$2N_1\cos\alpha + N_2 = P$$

Displacement compatibility:

$$\Delta l_1 = \Delta l_2\cos\alpha \qquad \frac{N_1 l}{EA} = \frac{N_2 l}{EA}\cos\alpha \qquad N_1 = N_2\cos\alpha$$

$$N_1 = \frac{P\cos\alpha}{1+2\cos^2\alpha} \le N_2 = \frac{P}{1+2\cos^2\alpha}$$

**Elastic limit capacity – plastic limit in bar #2**

$$N_2 = AR_e \qquad \overline{P} = AR_e\left(1+2\cos^2\alpha\right)$$

**Plastic limit capacity – plastic limit in bars #1 and #2**

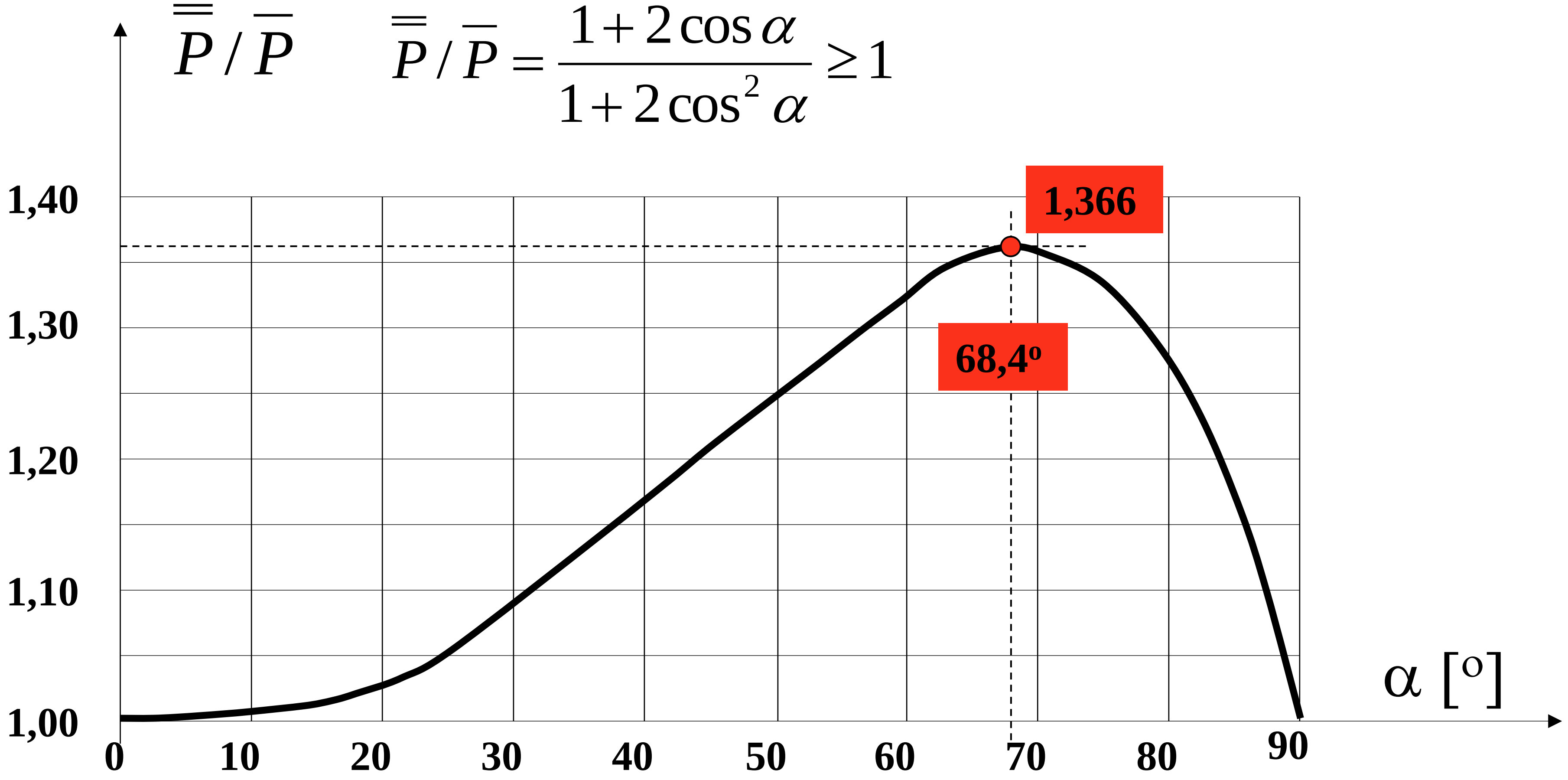
$$N_1 = N_2 = AR_e \qquad AR_e + 2AR_e\cos\alpha = \overline{\overline{P}} \qquad \overline{\overline{P}} = AR_e(1+2\cos\alpha)$$

$$\overline{\overline{P}}/\overline{P} = \frac{1+2\cos\alpha}{1+2\cos^2\alpha} \ge 1$$

# Limit analysis of structures – cont.

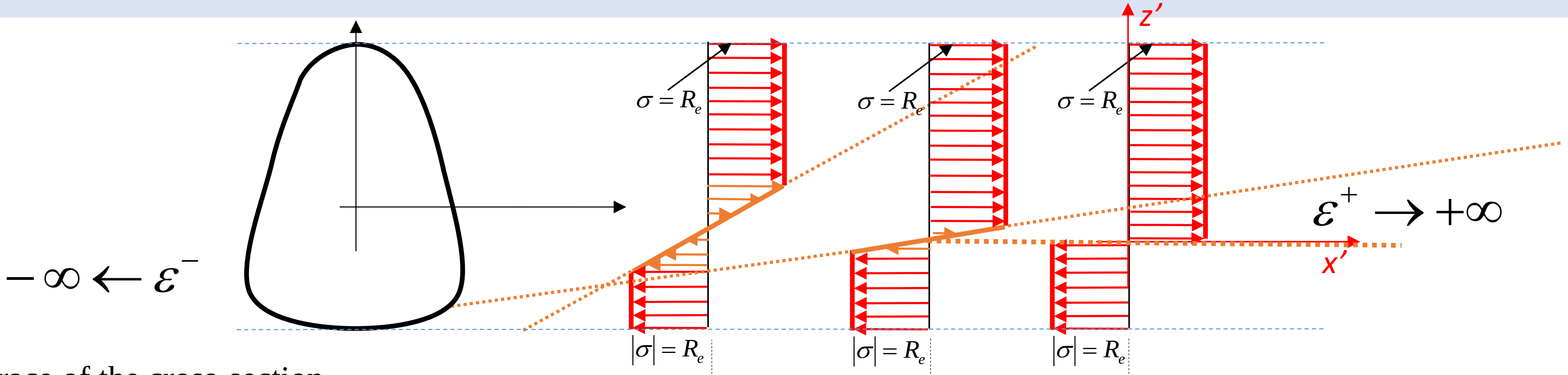
**Limit plastic capacity of 3-bars structure**

$$\overline{\overline{P}} / \overline{P} = \frac{1 + 2\cos\alpha}{1 + 2\cos^2\alpha} \geq 1$$



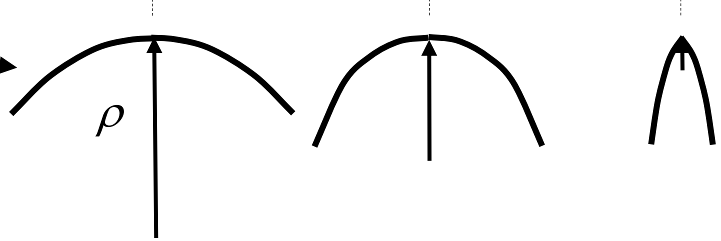

**Capacity of the 3-bar structure due to plastic properties**

# Limit analysis of bar structures – plastic hinge




Trace of the cross-section plane according to the Bernoulli hypothesis

Beam axis




$$\varepsilon_x = \frac{z}{\rho} = \kappa \cdot z$$

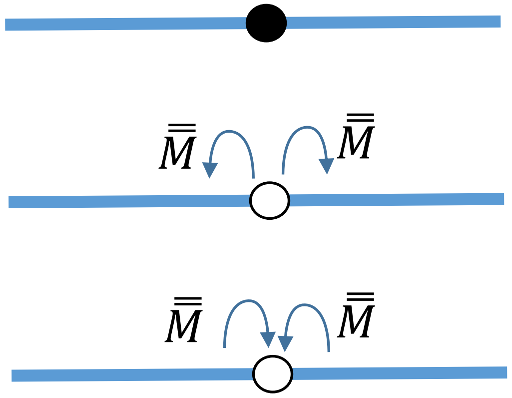
**Plastic hinge: $M = \bar{\bar{M}}$**

$$\varepsilon_x \rightarrow \infty \quad \Rightarrow \quad \kappa \rightarrow \infty \quad \Rightarrow \quad \rho \rightarrow 0$$

plastic hinge

tension at the bottom

tension at the top



# Plastic hinge – cont.

HEB 200 profile

steel $R_e = 350$ MPa, $\varepsilon_{\lim} = 0.3$, $E = 205$ MPa, $\varepsilon_{pl} = \frac{R_e}{E} = 1.707 \cdot 10^{-3}$

$\varepsilon(z = 0.1) = 0.3$, $\varepsilon_{pl}(z_p) = 1.707 \cdot 10^{-3} \rightarrow z_p = 0.000569$ mm

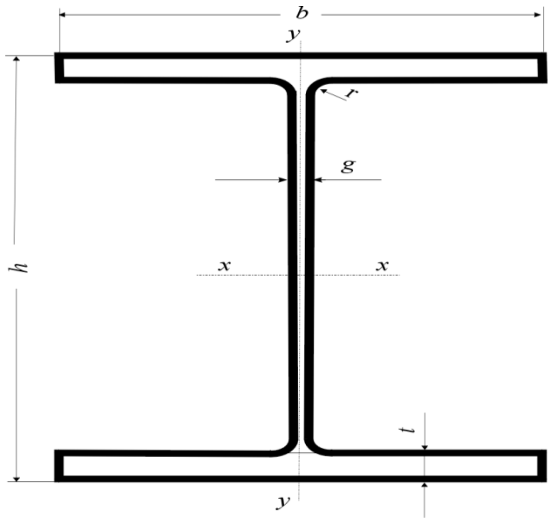
$\bar{\bar{M}} = 2(0.009 \cdot 0.085^2 \cdot 0.5 + 0.2 \cdot 0.015 \cdot 0.0925)R_e = 6.20010^{-4}\, R_e$

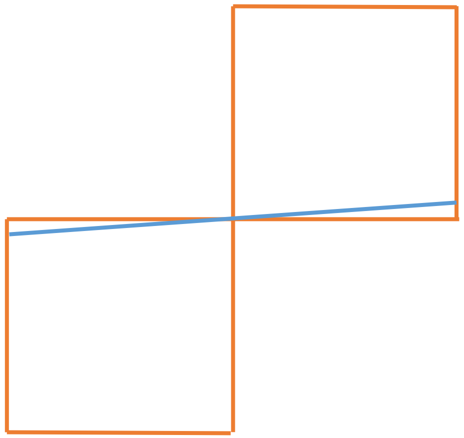
$M_{el} = 2R_e \cdot 0.009 \cdot 0.5 \cdot 0.000569^2 \cdot \frac{2}{3} = 1.943 \cdot 10^{-9} R_e$

$M_{el} = 0.0003\% \bar{\bar{M}}$

for a rectangular cross-section with $\varepsilon_{\lim} = 0.2$ and $\varepsilon_{pl} = 1.707 \cdot 10^{-3}$

$M_{el} = 0.00569\bar{\bar{M}}$



In the plasticity theory, it has been proved, that a structural damage doesn't depend on a change of elastic deformations.

When the plastic hinges has been developed, only plastic work (or dissipation energy) leads to the collapse of a structure.

A structure with the plastic hinges can be treated as a rigid between the hinges.

(see limit analysis of beams later on; the same mechanisms are considered in the yield line theory of plates)

# Moment – curvature interdependence

In elastic range:

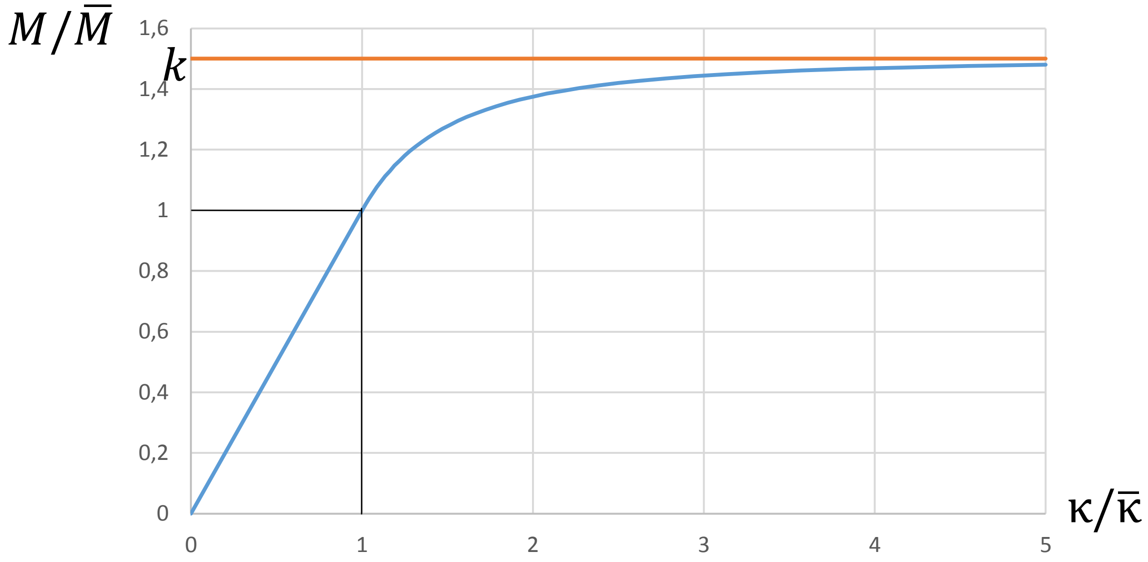
$$\frac{M}{EJ} = \frac{1}{\rho} = \kappa$$

$$M = \overline{M} \quad \Longrightarrow \quad \kappa = \overline{\kappa} = \frac{\overline{M}}{EJ}$$

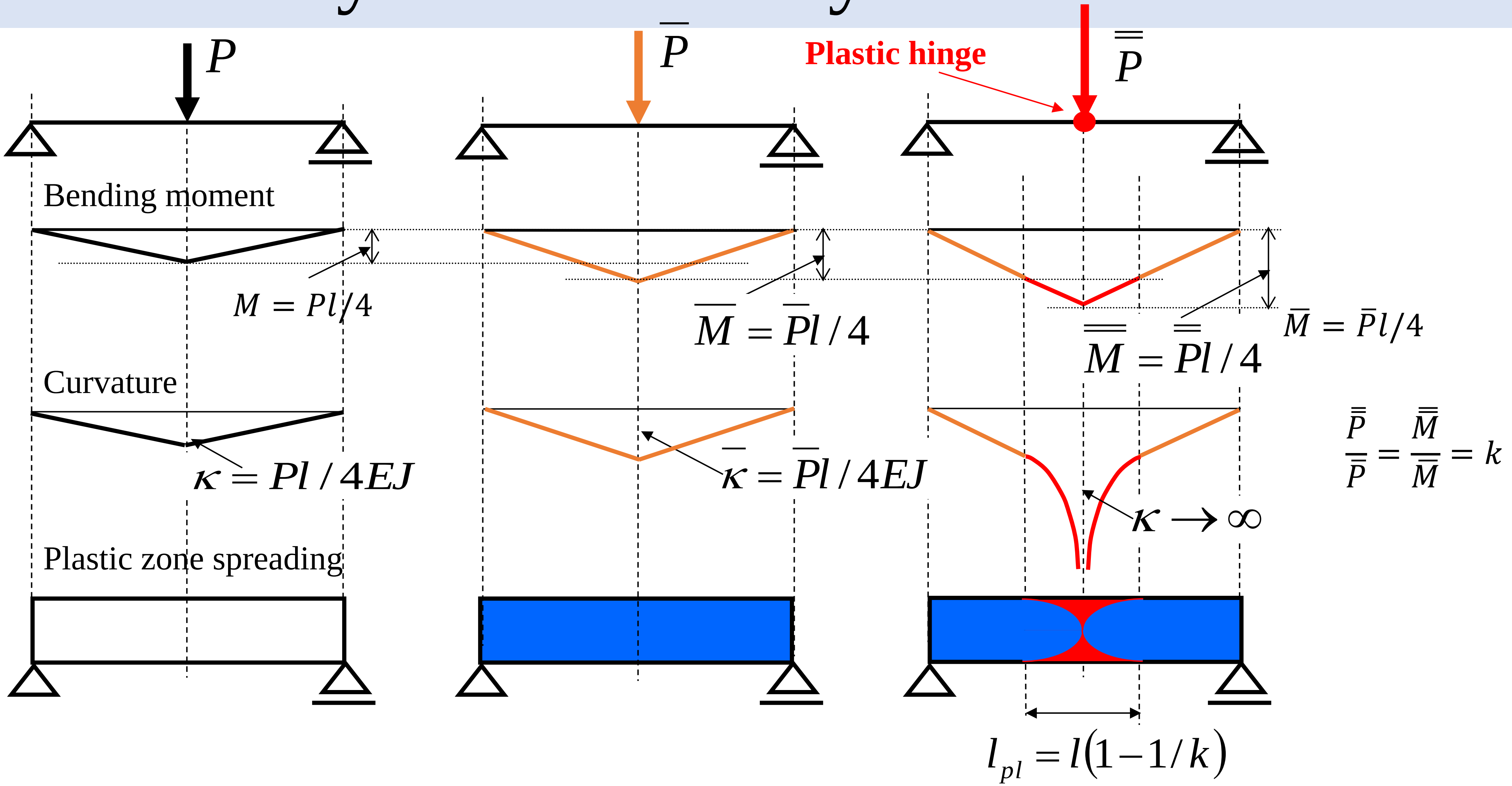
In plastic range:

$$M \overset{?}{=} f(\kappa)$$

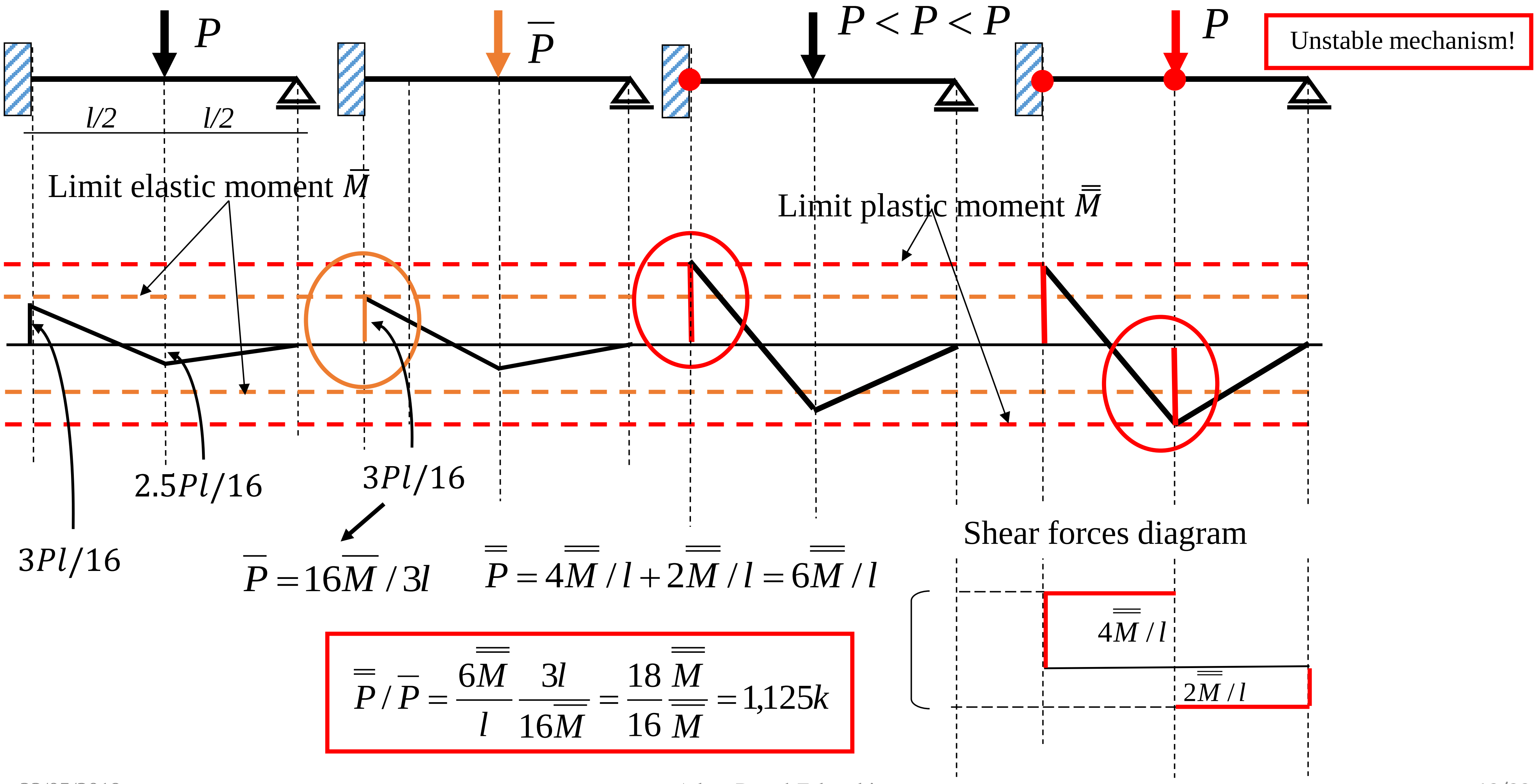
$$M = \overline{\overline{M}} \quad \Longrightarrow \quad \begin{cases} \overline{\overline{M}} / \overline{M} = k \\ \overline{\overline{\kappa}} \to \infty \end{cases}$$



# Limit analysis of statically determined beams



# Statically undetermined structures




$$\bar{P} = 16\bar{M}/3l \qquad \bar{\bar{P}} = 4\bar{\bar{M}}/l + 2\bar{\bar{M}}/l = 6\bar{\bar{M}}/l$$

$$\bar{\bar{P}}/\bar{P} = \frac{6\bar{\bar{M}}}{l}\frac{3l}{16\bar{M}} = \frac{18}{16}\frac{\bar{\bar{M}}}{\bar{M}} = 1{,}125k$$

# Limit analysis by virtual work principle

In limit plastic state the moment distribution due to given mechanism is known. Example:

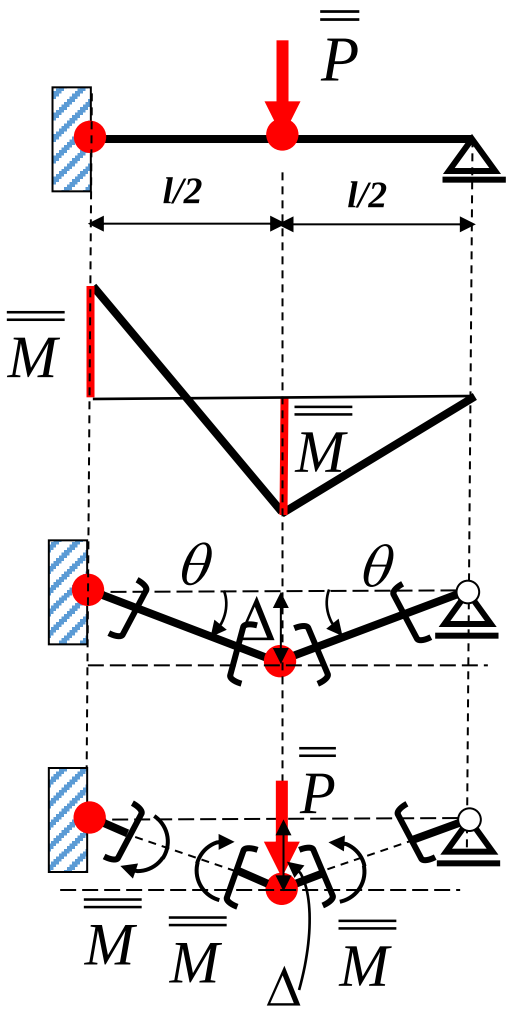
On this basis limit plastic capacity can be easily found, however, the ratio of plastic to elastic capacity is unavailable.

In a more complex case one has to consider all possible mechanisms. The right one is that which yields  the smallest value of limit plastic capacity.

$$3\overline{\overline{M}}\cdot\theta=\overline{\overline{P}}\cdot\theta\cdot l/2$$

$$\overline{\overline{M}}\cdot\theta+2\overline{\overline{M}}\cdot\theta=\overline{P}\cdot\Delta$$

$$\overline{\overline{P}}=6\overline{\overline{M}}/l$$

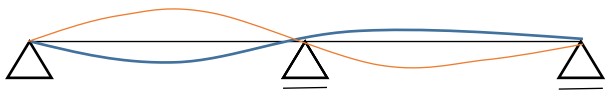


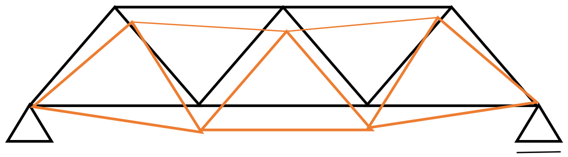
# Theorems of limit analysis – definitions

Statically admissible stress field – any stress field which:
- fulfills the equations of internal equilibrium,
- fulfills the static boundary conditions, and
- nowhere exceeds admissible stress value.

It means, for example, that everywhere:
- $|\sigma| < R_e$
- $|N| < \bar{\bar{N}}$
- $|M| < \bar{\bar{M}}$

Kinematically admissible field – any displacement field that:
- is compatible with the constraints
- fulfills the compatibility conditions, and
- fulfills condition of non-negative work of external forces.

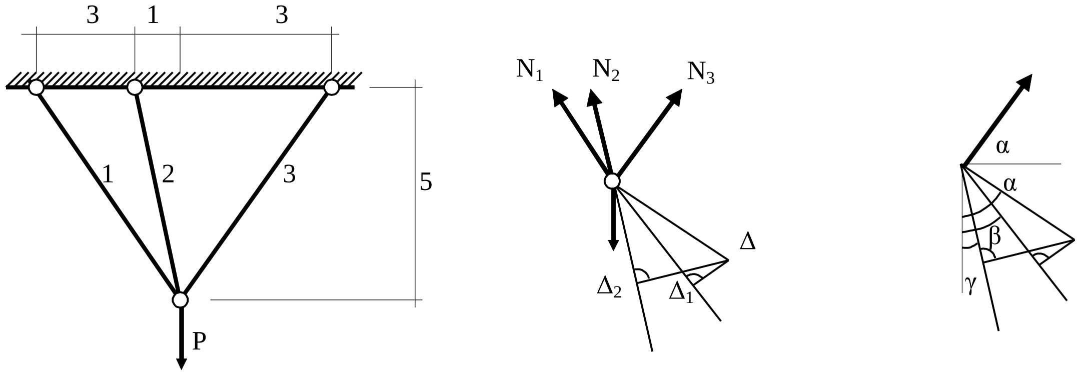




# Limit analysis – upper bound

The structure collapses (becomes a mechanism) or, at least is in limit equilibrium state if for kinematically admissible field of displacements the total work (power) of external forces is not less than the work (power) of internal forces.
In other words, if the structure collapses under external load, its bearing capacity is less or equal to the applied load.
This is the upper bound estimation.

Example
Find the limit bearing capacity of the truss below, $A_1 = 3\ \text{cm}^2, A_2 = 2\ \text{cm}^2, A_3 = 5\ \text{cm}^2\ , R_e = 400$ MPa.




Let's verify the scheme of yielding bars *1* and *2*.
The system has an instantaneous center of rotation at the end of the bar no 3.
From comparison of the work of external and internal forces, we get:

$$\mathbf{P} \cdot \boldsymbol{\Delta} = \bar{\bar{\mathbf{N}}}_1 \cdot \boldsymbol{\Delta}_1 + \bar{\bar{\mathbf{N}}}_2 \cdot \boldsymbol{\Delta}_2 \quad \rightarrow \quad P\Delta \frac{3}{\sqrt{34}} = 120\Delta_1 + 80\Delta_2$$

From the geometrical relations: $\alpha = \arccos \frac{3}{\sqrt{34}} = 59.04^0, \quad \beta = \arccos \frac{5}{\sqrt{34}} = 30.96^0, \quad \gamma = \arccos \frac{5}{\sqrt{26}} = 11.31^0$

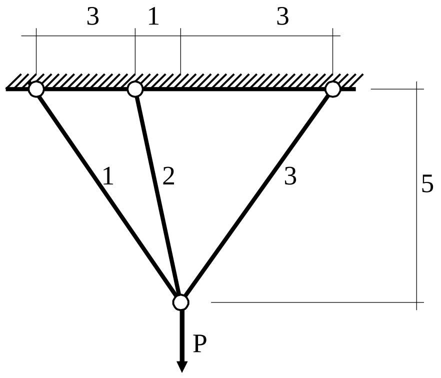
$\Delta_1 = \Delta \cos(\alpha - \beta) = 0.8823\Delta, \quad \Delta_2 = \Delta \cos(\alpha - \gamma) = 0.6726\Delta$ and finally: $\bar{\bar{P}} = 310.4$ kN

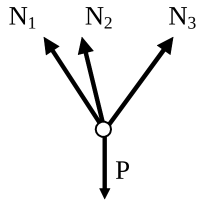
# Limit analysis – lower bound

The structure does not undergo destruction, or, at the most is in the state of limit equilibrium, if the statically admissible state of stress balances the actual loading.
In other words, the structure does not collapse if the external loading can be balanced by the statically admissible state of stress. The real bearing capacity is at least as the balanced load and it the value is lower or equal to the exact value.
This is the lower bound estimation.

If the lower bound estimation is equal to the upper bound estimation, the solution is exact to the real value of limit capacity.

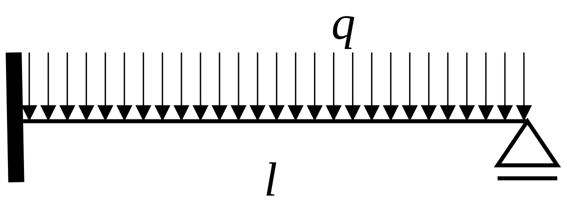
Example
Find the limit bearing capacity of the truss below, $A_1 = 3\ \text{cm}^2, A_2 = 2\ \text{cm}^2, A_3 = 5\ \text{cm}^2, R_e = 400$ MPa.

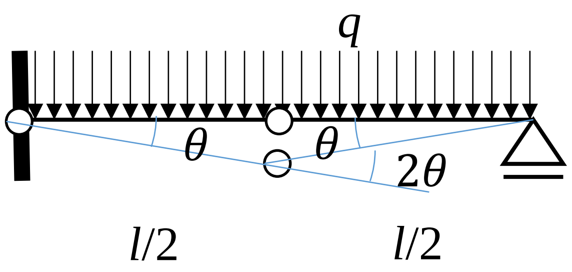



There is only one, statically admissible scheme:
$N_1 = \overline{\overline{N_1}} = 120$ kN, $N_2 = \overline{\overline{N_2}} = 180$ kN
$\Sigma X = 0 \rightarrow N_3 = 150.5\ \text{kN} < \overline{\overline{N_3}}$ (admissible)
$\Sigma Y = 0 \rightarrow \overline{\overline{P}} = 310.4$ kN
Other schemes:
$N_1 = \overline{\overline{N_1}}, N_3 = \overline{\overline{N_3}} \rightarrow N_2 > \overline{\overline{N_2}}$ (not admissible)
$N_2 = \overline{\overline{N_2}}, N_3 = \overline{\overline{N_3}} \rightarrow N_1 > \overline{\overline{N_1}}$ (not admissible)

# Upper and lower bounds – an example



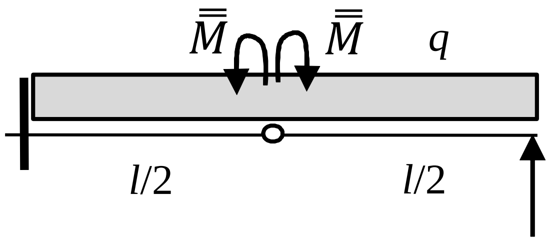



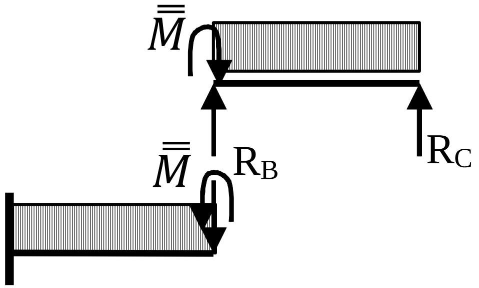

Kinematic scheme:

Virtual work:

$$2\int_0^{l/2} q\theta x dx = 3\theta\bar{\bar{M}} \;\rightarrow\; \bar{\bar{q}} = 12\frac{\bar{\bar{M}}}{l^2}$$

Static approach:




$$R_B = \frac{ql}{4} - 2\frac{\bar{\bar{M}}}{l}, \qquad M_u = \frac{ql^2}{4} - 2\bar{\bar{M}}$$

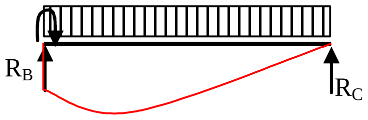
$$M_u = \bar{\bar{M}} \;\rightarrow\; \bar{\bar{q}} = 12\frac{\bar{\bar{M}}}{l^2}$$

Apparently, the upper and lower assessments are the same, but this is not a solution to the problem!
Why?

Because the reaction $R_B = 3\frac{\bar{\bar{M}}}{l} - 2\frac{\bar{\bar{M}}}{l} = \frac{\bar{\bar{M}}}{l}$>0, and it means that the shear force has zero-value point somewhere in the interval, and the bending moment has extreme value at the point and the value is greater than $\bar{\bar{M}}$ (that is statically not admissible).

# Example – contd.

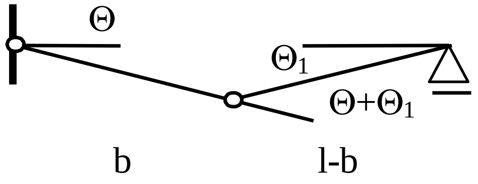
Let’s change the order of hinges creation and start wit a hinge at fixed end.




We calculate reaction $R_B$ and a shear force $Q(x)$ in the span. Searching a point of zero-value of the shear force and the extreme value of the bending moment, we assume there further plastic hinge getting an equation:

$$M(x_{\text{extr}}) = \overline{\overline{M}} \quad \rightarrow \quad \frac{\bar{\bar{q}}^2 l^2}{4} - 3\bar{\bar{q}}\,\overline{\overline{M}} + \frac{\overline{\overline{M}}^2}{l^2} = 0 \quad \text{with the solution:} \quad \bar{\bar{q}} = 11.66\frac{\overline{\overline{M}}}{l^2}$$

Similarly, we may determine the smallest value from the kinematic approach:
From the principle of virtual work:




$$\int_0^b q\Theta x dx + \int_0^{l-b} q\Theta_1 x_1 dx_1 = \overline{\overline{M}}(2\Theta + \Theta_1)$$

with $\Theta_1 = \dfrac{b}{l-b}\Theta$ we get: $q = \dfrac{2\overline{\overline{M}}}{bl}\dfrac{2l-b}{l-b}$

searching an extreme point: $\min \bar{\bar{q}} \quad \Rightarrow \quad \dfrac{\partial q}{\partial b} = 0 \quad \Rightarrow \quad b^2 - 4bl + 2l^2 = 0 \quad \Rightarrow \quad b = l\left(2-\sqrt{2}\right) = 0.59l$

finally, we get: $\bar{\bar{q}} = \dfrac{4}{6-4\sqrt{2}}\dfrac{\overline{\overline{M}}}{l^2} = 11.66\dfrac{\overline{\overline{M}}}{l^2}$ (the same value as in static approach, so this is the exact solution.)

# Residual stress

rectangular cross-section

elastic and plastic limit moments: $\bar{M} = \frac{bh^2}{6} R_e$, $\bar{\bar{M}} = \frac{bh^2}{4} R_e$

applied elastic-plastic moment: $M = \bar{M} + \alpha\left(\bar{\bar{M}} - \bar{M}\right)$

$$\alpha = 0.5 \rightarrow M = \frac{5bh^2}{24} R_e$$

$$M = 2\left[b\int_0^{\xi} E\kappa z^2\,dz + b\int_{\xi}^{h/2} R_e z\,dz\right], \quad E\kappa\xi = R_e$$

$$M = \frac{bR_e}{3}\left(\frac{3}{4}h^2 - \xi^2\right) \rightarrow \xi = \frac{h}{2\sqrt{2}} = 0.35355h$$

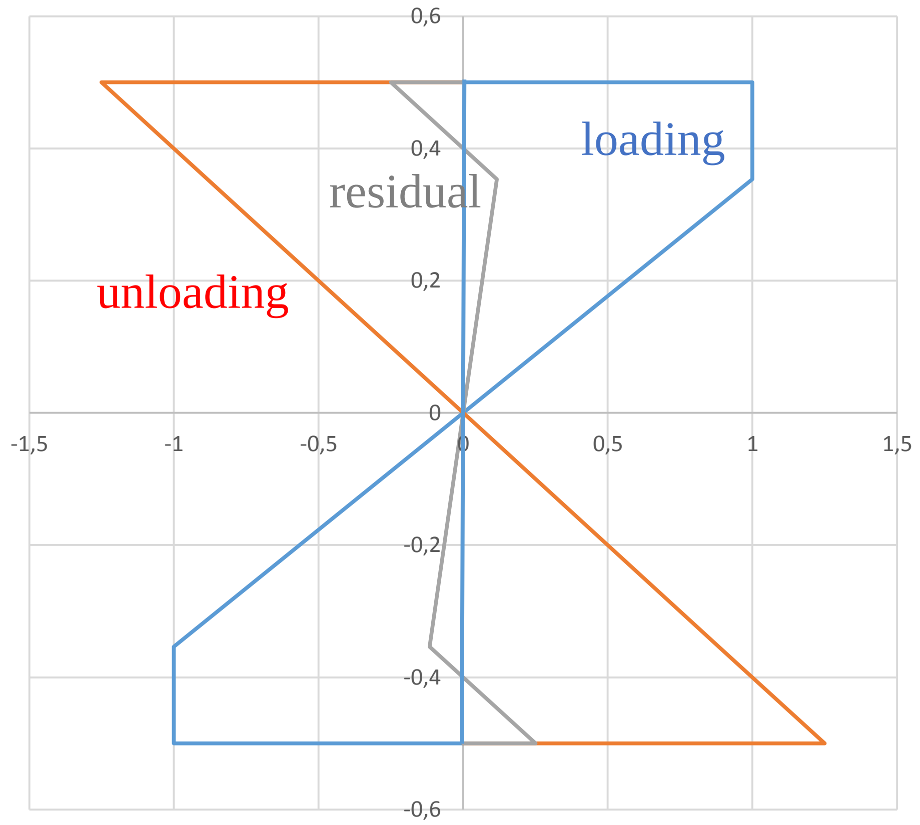
unloading is an elastic proces, so:

$$\sigma_{\max} = \sigma(0.5h) = \frac{M}{\bar{W}} = \frac{5bh^2 R_e \cdot 6}{bh^2} = \frac{5}{4} R_e = 1.25 R_e$$

$$\sigma(0.35355h) = \frac{0.35355}{0.5} 1.25 R_e = 0.8839 R_e$$

finally:

$$\sigma(0.5h) = R_e - 1.25 R_e = -0.25 R_e$$
$$\sigma(0.35355h) = R_e - 0.8839 R_e = 0.1161 R_e$$



# Residual strain and curvature

In uniaxial tension there is no residual stress because the problem is statically determined. There are permanent strains only, that remain after unloading.
The residual stress remains in the statically undetermined cases only.

The residual curvature calculation
plastic loading curvature:

$$E\kappa 0.35355h = R_e \;\rightarrow\; \kappa_{pl} = \frac{350}{205000\cdot 0.35355h} = \frac{4.829\cdot 10^{-3}}{h}\;[1/\mathrm{m}]$$

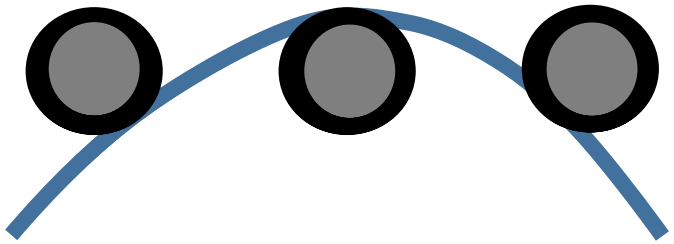
elastic unloading curvature:

$$E\kappa \frac{h}{2} = 1.25R_e \;\rightarrow\; \kappa_{\mathrm{el}} = \frac{1.25\mathrm{R_e}}{\mathrm{E}\frac{\mathrm{h}}{2}} = \frac{1.25\cdot 350}{205000}\cdot\frac{2}{h} = \frac{4.268\cdot 10^{-3}}{h}\;[1/\mathrm{m}]$$

residual curvature:

$$\kappa_{pl} - \kappa_{el} = 5.61\cdot 10^{-4}[1/\mathrm{m}], \rightarrow\; \rho_{\mathrm{res}} = 1.78\;[\mathrm{m}]$$

realignment of wire
(bar bending machine):



An inverse problem:
What bending moment should be applied to realign the considered bar with the initial curvature $\rho = 1.78$ m?

# Interaction curves

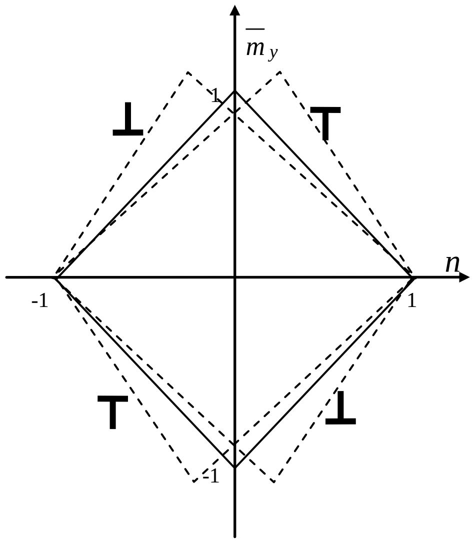
Interaction curves – the curves in the space of cross-sectional forces which correspond to a particular mechanical state (bearing capacity: limit elastic or limit plastic).

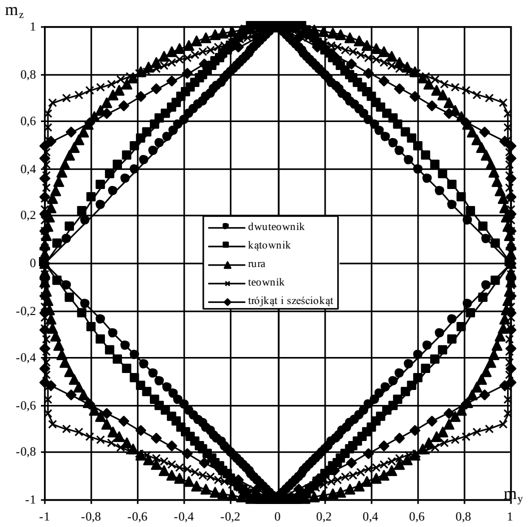
Dimensionless cross-sectional forces:

$$\bar{n} \equiv \frac{N}{\bar{\bar{N}}}, \qquad \bar{\bar{n}} \equiv \frac{N}{\bar{\bar{\bar{N}}}}, \qquad \bar{m}_y \equiv \frac{M_y}{\bar{\bar{M}}_y}, \qquad \bar{\bar{m}}_y \equiv \frac{M_y}{\bar{\bar{\bar{M}}}_y}, \qquad \bar{m}_z \equiv \frac{M_z}{\bar{\bar{M}}_z}, \qquad \bar{\bar{m}}_z \equiv \frac{M_z}{\bar{\bar{\bar{M}}}_z}$$

elastic limit state:

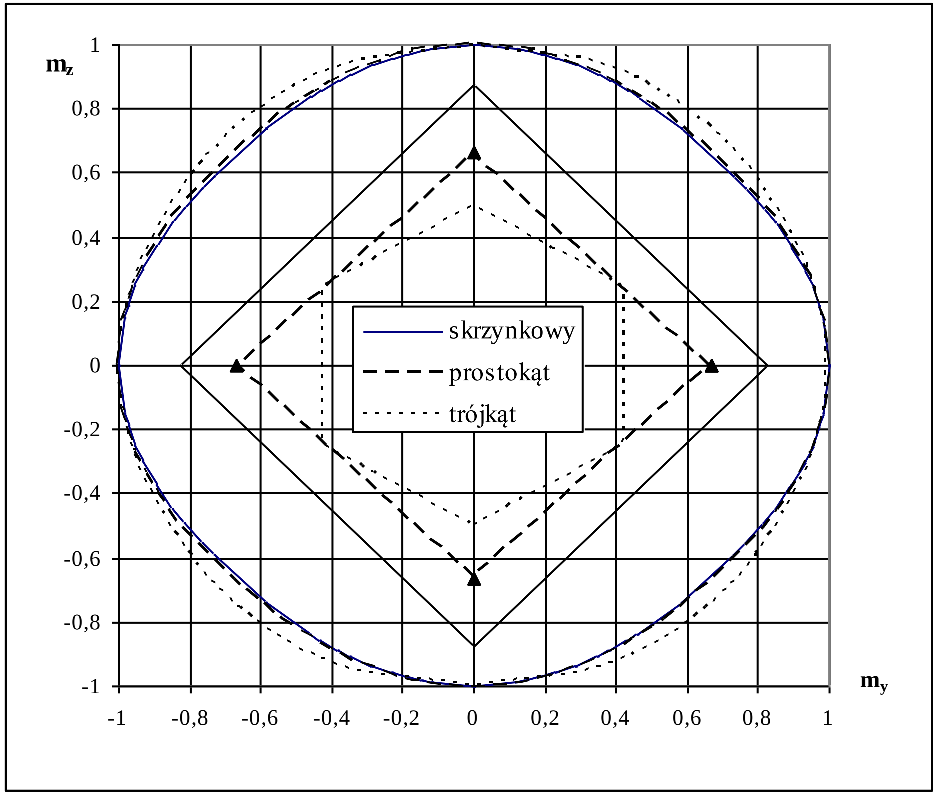
$\max|\sigma| = R_e \wedge A_{pl} = 0$



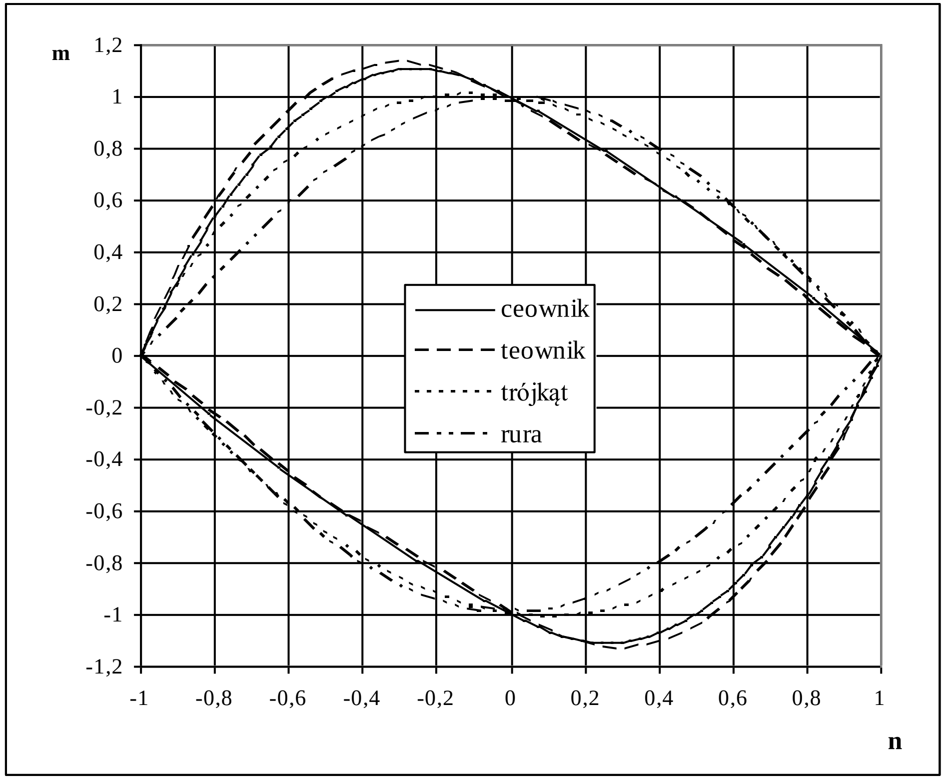




I beam
angle
tube
T shape
triangle & hexagon

# Interaction curves – cont.




box
rectangle
triangle




C-channel
T-shape
triangle
tube

# Thank you for your attention!