Strength of Materials

12. Fracture

Stress concentration

Relatively simple laws may be in error due to:

- abrupt changes in section (the threads of a bolt, a plate containing a hole, at the corner of a keyway in a shaft)
- contact stress at the point of application of the external forces (the wheels of a locomotive and the rail, between gear teeth, or between ball bearings and their races)
- discontinuities in the material itself (nonmetallic inclusions in steel, voids in concrete, pitch pockets and knots in timber, or variation in strength and stiffness of the element, such as fibers in wood, aggregate in concrete)
- initial stress in a member resulting from fabrication, from heat treatment, from shrinkage in concrete, or from residual stress resulting from welding
- cracks that exist in the member resulting from fabrication (welding, cold working, grinding, etc.)
- loading which changes in time
- material properties which change in time

The discontinuities or stress raisers cause sudden increases in the stress (stress peaks) at points near the stress raisers.

Usually, large stresses resulting from discontinuities are developed in only a small portion of a member. These stresses are called *localized stresses* or *stress concentrations*. The rate of increase of stress is indicated by *stress gradient*.

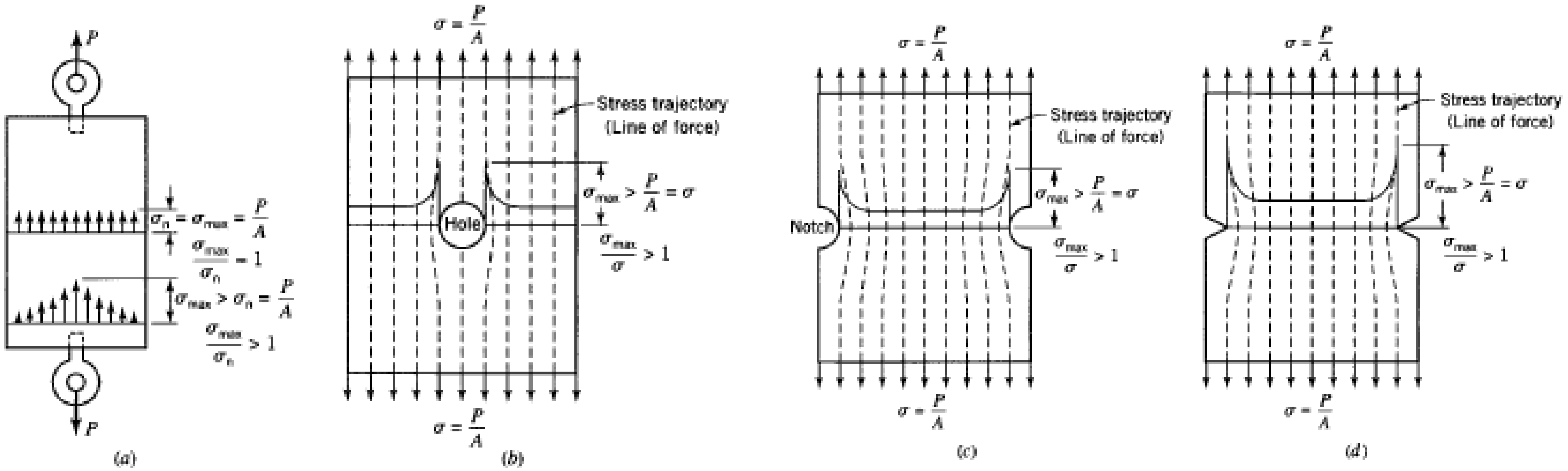
To obtain an estimate of the local stress at the point, the concept of a *stress concentration factor* is often employed.

Stress concentration factor

In the tension test the stress state is uniformly distributed over the cross-section. The stress raisers cause not uniform stress distribution with a maximum stress considerably larger that the average stress. The ratio:

$$S_c = \frac{\sigma_{\max}}{\sigma_n}$$

is called the *stress concentration factor* (or, sometimes, *strength reduction factor*).

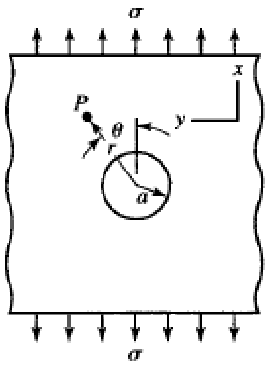


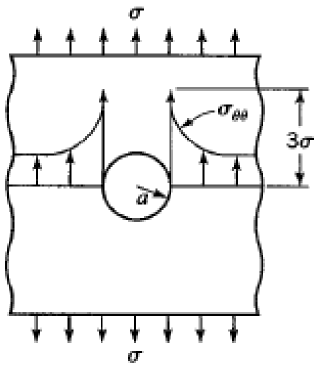
Stress concentration factor – circular hole

Let’s consider an infinite plate with a small circular hole of radius a under uniaxial tension σ.
With respect to polar coordinates (r, θ), the circumferential stress at the hole boundary is given by:

$$\sigma_{\theta\theta} = \sigma(1 - 2\cos 2\theta)$$

and attains its maximum value of $\max\sigma_{\theta\theta} = 3\sigma$ for $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$. For $\theta = 0$ and $\theta = \pi \rightarrow \ \sigma_{\theta\theta} = -\sigma$ (compression).



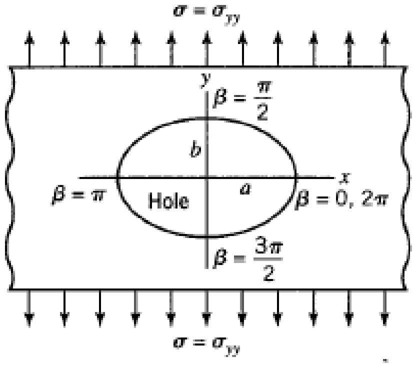


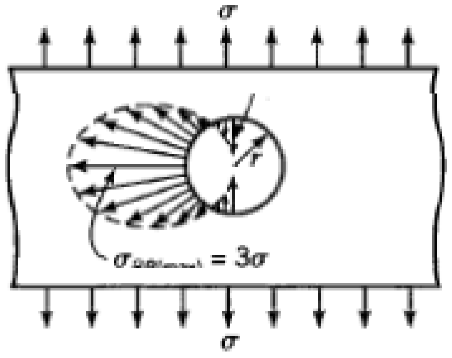
The results are summarized by:

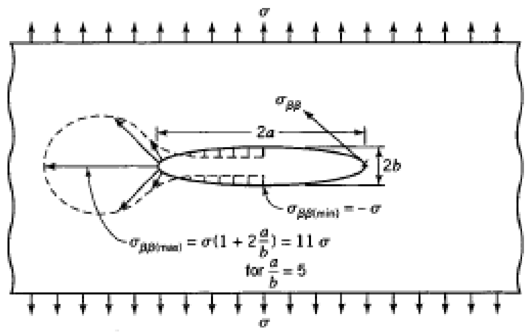
$$S_c = \frac{\sigma_{\max}}{\sigma_n} = \frac{3k - 1}{k + 0.3}$$

where k is the ratio (width of strip)/(diameter of hole) and σ_n is the average stress over the weakened cross-sectional area.

Stress concentration factor – elliptical hole







For the strip with an elliptic hole, the stress distribution has been determined by Inglis (1913):

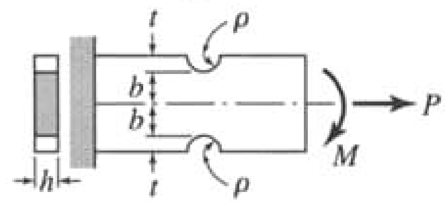
$$\sigma_{\beta\beta(\max)} = \sigma(1 + 2\coth\alpha_0) = \sigma\left(1 + \frac{2a}{b}\right)$$

When the elliptical hole becomes a sharp crack (an ellipse of zero height and length $2a$), $\sigma_{\beta\beta}$ increases without bond as $b/a \to 0$.

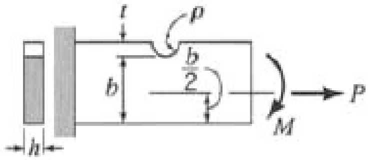
For an ellipsoidal (3D) cavity, the stress concentration factors are given in the table below.

Ratio b/a	**1.0**	**0.8**	**0.6**	**0.4**	**0.2**	**0.1**
Prolate spheroid shape	2.05	2.17	2.33	2.52	2.70	2.83
Oblate spheroid shape	2.05	2.50	3.3	4.0	7.2	13.5

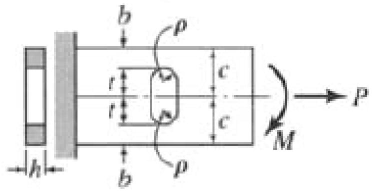
Stress concentration – grooves and holes



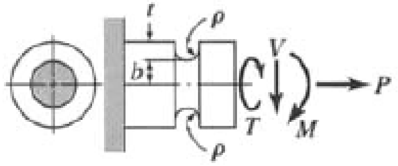
symmetric grooves



one-side groove



symmetric hole



circular groove

semi-empirical formulae from Neuber (1958)

- shallow groove

$$S_{cs} = 1 + 2\sqrt{\frac{t}{\rho}}$$

- deep groove

$$S_{cd} = \frac{2\left[\left(\frac{b}{\rho}\right) + 1\right]\sqrt{\frac{b}{\rho}}}{\left[\left(\frac{b}{\rho}\right) + 1\right]\tan^{-1}\sqrt{\frac{b}{\rho}} + \sqrt{\frac{b}{\rho}}}$$

- moderate groove

$$S_c = 1 + \frac{(S_{cs} - 1)(S_{cd} - 1)}{\sqrt{(S_{cs} - 1)^2 + (S_{cd} - 1)^2}}$$

Stress concentration – elliptical hole

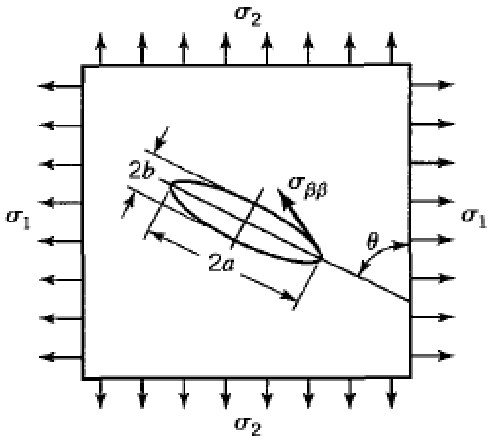
Infinite plate with an elliptical hole, uniformly stressed along:

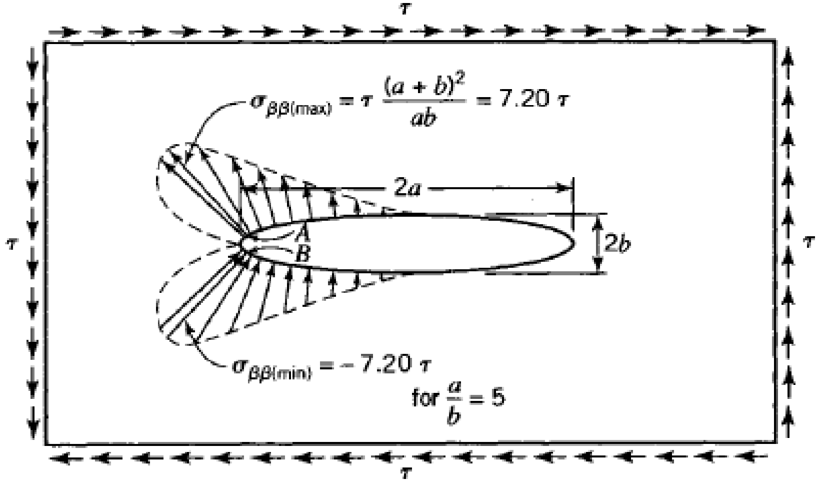
- both ellipse axes directions

$$\sigma_{\alpha\alpha} + \sigma_{\beta\beta} = \frac{2\sigma \sinh 2\alpha}{\cosh 2\alpha - \cosh 2\beta} \rightarrow \sigma_{\beta\beta(\max)} = 2\sigma\left(\frac{a}{b}\right)$$

- two perpendicular directions

$$\sigma_{\beta\beta} = [(\sigma_1 + \sigma_2)\sinh 2\alpha_0 + (\sigma_1 - \sigma_2)(e^{2\alpha_0}\cos 2\beta - 1)\cos 2\theta$$
$$(\sigma_1 - \sigma_2)e^{2\alpha_0}\sin 2\beta \sin 2\theta]/(\cosh 2\alpha_0 - \cos 2\beta)$$





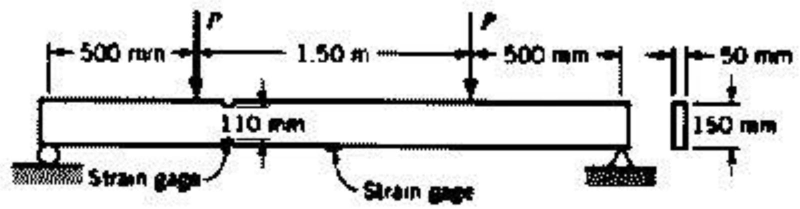
Pure shear parallel to the axes of the elliptical hole

$$\sigma_{\beta\beta}\Big|_{\alpha=\alpha_0} = \frac{2\tau e^{2\alpha_0}\sin 2\beta}{\cosh 2\alpha_0 - \cos 2\beta}$$

$$\sigma_{\beta\beta(\max)} = \tau\frac{(a+b)^2}{ab}$$

Stress concentration – an example

The rectangular section beam is made of an aluminum alloy (E =72 GPa). Strain-gage readings at the bottom of the grove and at some distance from the stress concentration were recorded as 0.00250 and 0.00100, respectively. Determine the magnitudes of P and the stress concentration factor S_c.



Solution

The bending moment in the span between the points of the forces application is constant, $M = Pa$.

At the point distant from the groove:

$$\max\sigma_x = \frac{\mathrm{M}}{\mathrm{W}} = \frac{6\mathrm{Pa}}{\mathrm{bh}^2} = \frac{6 \cdot 0.5 \cdot \mathrm{P}}{0.05 \cdot 0.15^2} = 2666.67\mathrm{P}$$

from the measured strain: $\varepsilon = 0.001 \; \rightarrow \; \sigma = E\varepsilon = 72 \cdot 10^9 \cdot 0.01 = 72$ MPa, so $P = 27$ kN. For this value of forces, we get at the groove cross-section:

$$\sigma_{xg} = \frac{M}{W_g} = \frac{6Pa}{bh_g^2} = \frac{6 \cdot 0.5 \cdot 27 \cdot 10^3}{0.05 \cdot 0.15^2} = 133.9\; MPa \; \rightarrow \; \varepsilon_g = \frac{\sigma_{xg}}{E} = \frac{133.9 \cdot 10^3}{27 \cdot 10^9} = 0.00186$$

The measured value of the stress at the groove is: $\sigma_{gm} = E\varepsilon = 0.0025 * 72 \cdot 10^9 = 180$ MPa.

Finally, the stress concentration factor is: $S\_c = 180/133.9 = 1.344$

Elliptical hole – an example

Consider an elliptical hole in a plate with ratio $a/b = 100$ (very narrow slit/crack). Let compressive stresses $\sigma_1 = -20$ MPa and $\sigma_2 = -75$ MPa be applied to the plate edges. (a) Determine the orientation of the hole (value of θ) for which the tensile stress at the perimeter of the hole is a maximum. (b) Calculate the value of this tensile stress. (c) Calculate the associated value of β (location of the point) for which this tensile stress occurs.

Solution:

The formula 5_1 indicates that $\coth \alpha_0 = a/b = 100$. Hence, $\alpha_0 = 0.01$rad, $\sinh 2\alpha_0 = 0.02, \cosh 2\alpha_0 = 50.0$.

(a) The value of θ is given by Eq. 7_2:

$$\cos 2\theta = \left[\frac{-1 - 0.02 + \left(\frac{-20-75}{-20+75}\right)^2 (0.02)}{2\left(\frac{-20-75}{-20+75}\right)}\right] = 0.2607 \rightarrow \theta = 0.6535 \text{ rad} = 37.44°$$

(b) The maximum value of the tensile stress can be found from Eq. 7_2:

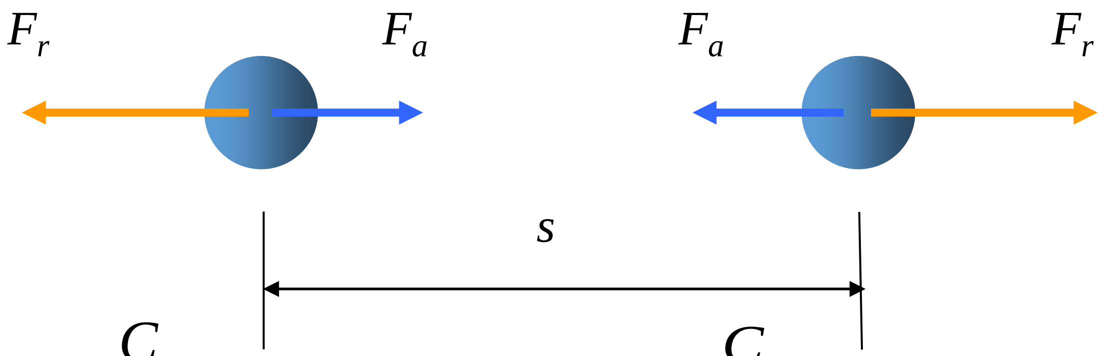
$$\sigma_{\beta\beta(\max)} = \sigma_{\beta\beta 1} = \frac{(-20+75)^2}{2(-20+75)}(1+50) = 812 \text{ MPa (tension)}$$

(c) This tensile stress is located on the perimeter of the hole at a value of β given by Eq. 7_2

$$\cos 2\beta = 1 - 2\left(\frac{-20-75}{-20+75}\right)^2 \frac{0.02^2}{1.02} = 0.9977 \rightarrow \beta = 0.0342 \text{ rad} = 1.96°$$

(the maximum tensile stress occurs very near the end of the major axis) (the example from: *Boresi*)

Fracture – Lennard-Jones model



$$F_r = \frac{C_r}{s^m}$$

$$F_a = \frac{C_a}{s^n}$$

$m > n$ $(m \approx 12,\ n \approx 6)$

Reactive force (repulsion)

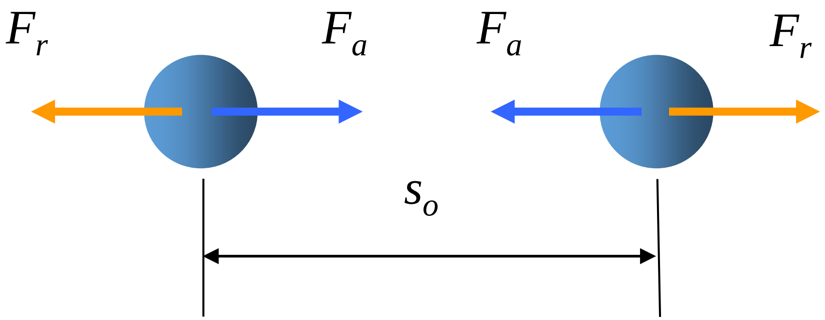
Active force (attraction)

For $s = s_o$

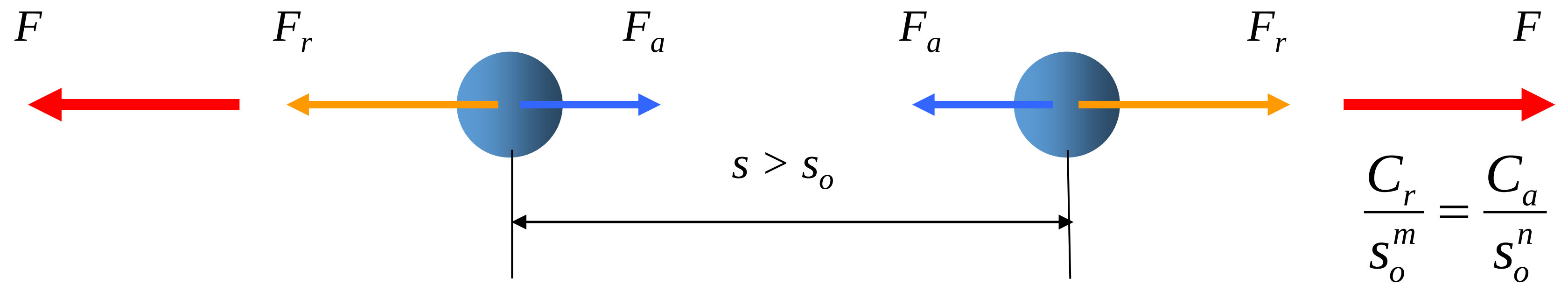
$$F_a = F_r \Rightarrow \frac{C_a}{s_o^n} = \frac{C_r}{s_o^m}$$

$$\Downarrow$$

$$\frac{C_a}{C_r} = \frac{s_o^n}{s_o^m} = s_o^{n-m} = \frac{1}{s_o^{m-n}}$$

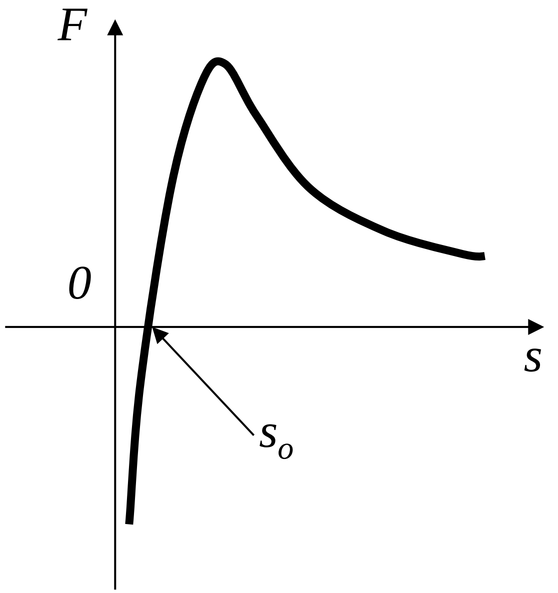


Lennard-Jones model – cont.

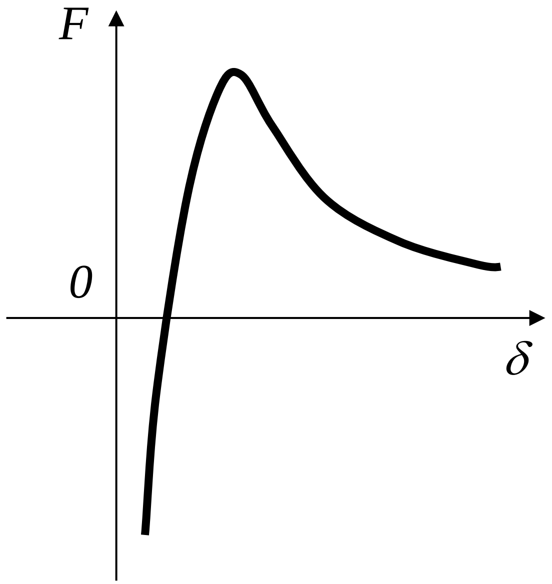


$$\frac{C_r}{s_o^m} = \frac{C_a}{s_o^n}$$

$$F = F_a - F_r = \frac{C_a}{s^n} - \frac{C_r}{s^m} = \frac{C_a}{s_o^n}\frac{s_o^n}{s^n} - \frac{C_r}{s_o^m}\frac{s_o^m}{s^m} = \frac{C_a}{s_o^n}\left[\left(\frac{s_o^n}{s^n}\right) - \left(\frac{s_o^m}{s^m}\right)\right]$$



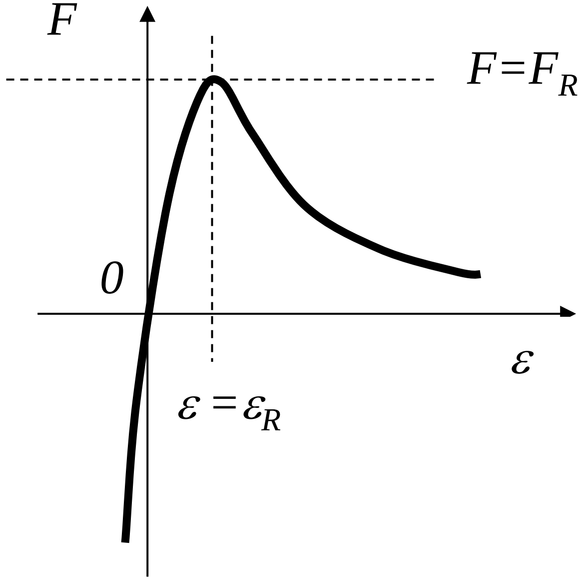
$$\delta = s - s_o$$



Lennard-Jones model – cont.

$$F=\frac{C_a}{s_o^n}\left[\left(\frac{s_o^n}{s^n}\right)-\left(\frac{s_o^m}{s^m}\right)\right]=\frac{C_a}{s_o^n}\left[\left(\frac{s_o}{s}\right)^n-\left(\frac{s_o}{s}\right)^m\right]=\frac{C_a}{s_o^n}\left[\left(\frac{s_o}{s_o+\delta}\right)^n-\left(\frac{s_o}{s_o+\delta}\right)^m\right]$$

$$F=\frac{C_a}{s_o^n}\left[\left(\frac{1}{1+\delta/s_o}\right)^n-\left(\frac{1}{1+\delta/s_o}\right)^m\right]=\frac{C_a}{s_o^n}\left[\left(\frac{1}{1+\varepsilon}\right)^n-\left(\frac{1}{1+\varepsilon}\right)^m\right]=\frac{C_a}{s_o^n}\left[(1+\varepsilon)^{-n}-(1+\varepsilon)^{-m}\right]$$



$$\varepsilon=\frac{\delta}{s_o}$$

$$\left.\frac{dF}{d\varepsilon}\right|_{\varepsilon_R}=0$$

$$\frac{C_a}{s_o^n}\left.\frac{dF}{d\varepsilon}\right|_{\varepsilon_R}=-n(1+\varepsilon_R)^{-n-1}-(-m)(1+\varepsilon_R)^{-m-1}=0$$

$$(1+\varepsilon_R)^{-n-1+m+1}=\frac{m}{n}\qquad \varepsilon_R=\left(\frac{m}{n}\right)^{\frac{1}{m-n}}-1$$

Lennard-Jones model – cont.

$$\varepsilon_R = \left(\frac{m}{n}\right)^{\frac{1}{m-n}} - 1 \quad \Rightarrow \quad F_R = \frac{C_a}{s_o^n}\left[\left(\frac{m}{n}\right)^{\frac{-n}{m-n}} - \left(\frac{m}{n}\right)^{\frac{-m}{m-n}}\right]$$

m=12, n =6

$$\varepsilon_R = \left(\frac{12}{6}\right)^{\frac{1}{12-6}} - 1 = 2^{\frac{1}{6}} - 1 \cong 0{,}112 = 11.2\%$$

$$F_R = \frac{C_a}{s_o^n}\left[\left(\frac{12}{6}\right)^{\frac{-6}{12-6}} - \left(\frac{12}{6}\right)^{\frac{-12}{12-6}}\right] = \frac{C_a}{s_o^n}\left[2^{-1} - 2^{-2}\right] = 0{,}25\frac{C_a}{s_o^n} = (10 \div 1000)\ (F_{R\,\mathrm{experimental}})$$

Reasons for discrepancy:

1. Extremely simplified two-atomic model
2. Defects of crystalline structure (theory of dislocation)

Fatigue – definition

Fatigue has been defined as „the progressive localized permanent structural change that occurs in a material subjected to repeated or fluctuating strains at stresses having a maximum value less than the tensile strength of the material”.

It has been estimated that between 50% and 90% of failures are due to fatigue.

The *fatigue life* of a member is the number of load cycles or the time during which the member is subjected to these loads before fracture occurs.

The fatigue life is affected by:

- the type of load (uniaxial, bending, torsion)
- the nature of the load-displacement curve (linear, nonlinear)
- the frequency of load repetitions or cycling
- the load history (constant or variable amplitude, random load, etc.)
- the size of the member
- the material flaws
- the manufacturing method
- the operating temperatures
- the environmental operating conditions (corrosion, etc.)

Fatigue – characteristics

- In metal alloys, and for the simplifying case when there are no macroscopic or microscopic discontinuities, the process starts with dislocation movements at the microscopic level, which eventually form persistent slip bands that become the nucleus of short cracks
- Macroscopic and microscopic discontinuities (at the crystalline grain scale) as well as component design features which cause stress concentrations (holes, keyways, sharp changes of load direction etc.) are common locations at which the fatigue process begins
- Fatigue is a process that has a degree of randomness (stochastic), often showing considerable scatter even in seemingly identical sample in well controlled environments
- Fatigue is usually associated with tensile stresses but fatigue cracks have been reported due to compressive loads
- The greater the applied stress range, the shorter the life
- Fatigue life scatter tends to increase for longer fatigue lives
- Damage is cumulative; materials do not recover when rested
- Fatigue life is influenced by a variety of factors, such as temperature, surface finish, metallurgical microstructure, presence of oxidizing or inert chemicals, residual stresses, scuffing contact (fretting), etc.
- Some materials (e.g., some steel and titanium alloys) exhibit a theoretical fatigue limit below which continued loading does not lead to fatigue failure

Fatigue – characteristics cont.

- High cycle fatigue strength (about 10^4 to 10^8 cycles) is examined with resonant magnetic machines
- Low cycle fatigue (loading that typically causes failure in less than 10^4 cycles) is associated with localized plastic behavior in metals.

Design against fatigue

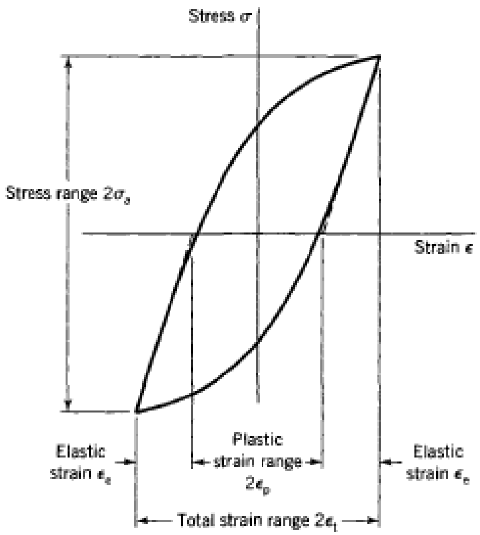
1. Design to keep stress below threshold of fatigue limit (infinite lifetime concept)
2. Fail-safe, graceful degradation, and fault-tolerant design: Instruct the user to replace parts when they fail. Design in such a way that there is no single point of failure, and so that when any one part completely fails, it does not lead to catastrophic failure of the entire system
3. Safe-life design: Design (conservatively) for a fixed life after which the user is instructed to replace the part with a new one (a so-called *lifed* part, finite lifetime concept, or "safe-life" design practice); planned obsolescence and disposable product are variants that design for a fixed life after which the user is instructed to replace the entire device
4. Damage tolerant design: Instruct the user to inspect the part periodically for cracks and to replace the part once a crack exceeds a critical length. This approach usually uses the technologies of nondestructive testing and requires an accurate prediction of the rate of crack-growth between inspections. The designer sets some aircraft maintenance checks schedule frequent enough that parts are replaced while the crack is still in the "slow growth" phase. This is often referred to as damage tolerant design or "retirement-for-cause".

Fatigue – phases

The fatigue life (total life) may be considered to consist of three phases:

1. Initial fatigue damage that produces crack initiation
2. propagation of a crack or cracks that result in partial separation of a cross-section of a member
3. final fracture of the member.

The microscopic cracks develop very early in the fatigue life and grow at various rates until fracture occurs.
The material that undergo low cycle fatigue is considered to be a ductile material. Fatigue failure may occur brittle resulting from high cycle fatigue. For a cycles number $N < 10^5$ the fatigue is usually referred to as low cycle fatigue data.



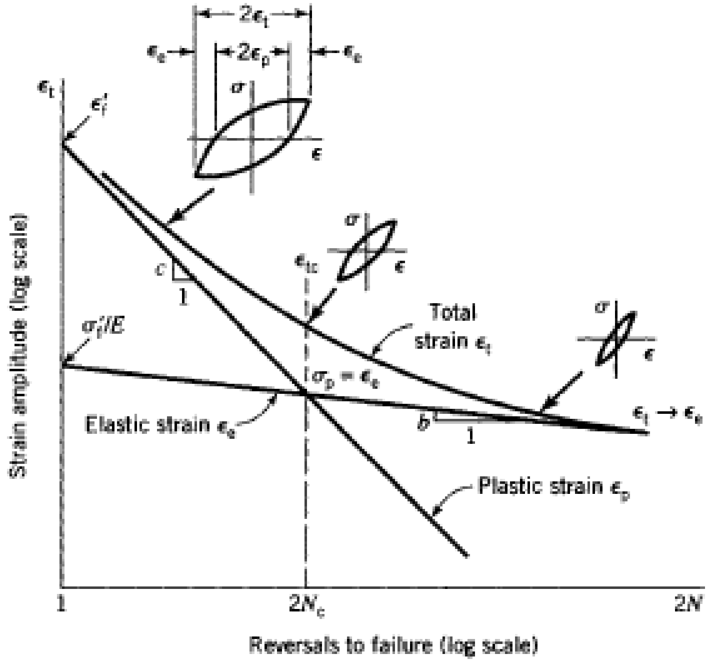
Low cycle fatigue
The final state of a stabilized stress-strain hysteresis loop
The area inside the hysteresis loop is the energy per unit volume dissipated during a cycle, usually in the form of heat.

Low cycle fatigue

The fatigue life is the number of cycles with constant strain amplitude that can be sustained by a specimen. The total strain amplitude consists of two parts: the plastic strain amplitude and elastic strain amplitude.

At a certain life, the elastic and plastic strains are equal. It is sometimes called the transitional or cross-over number of cycles.
Three different failure criteria have been used:

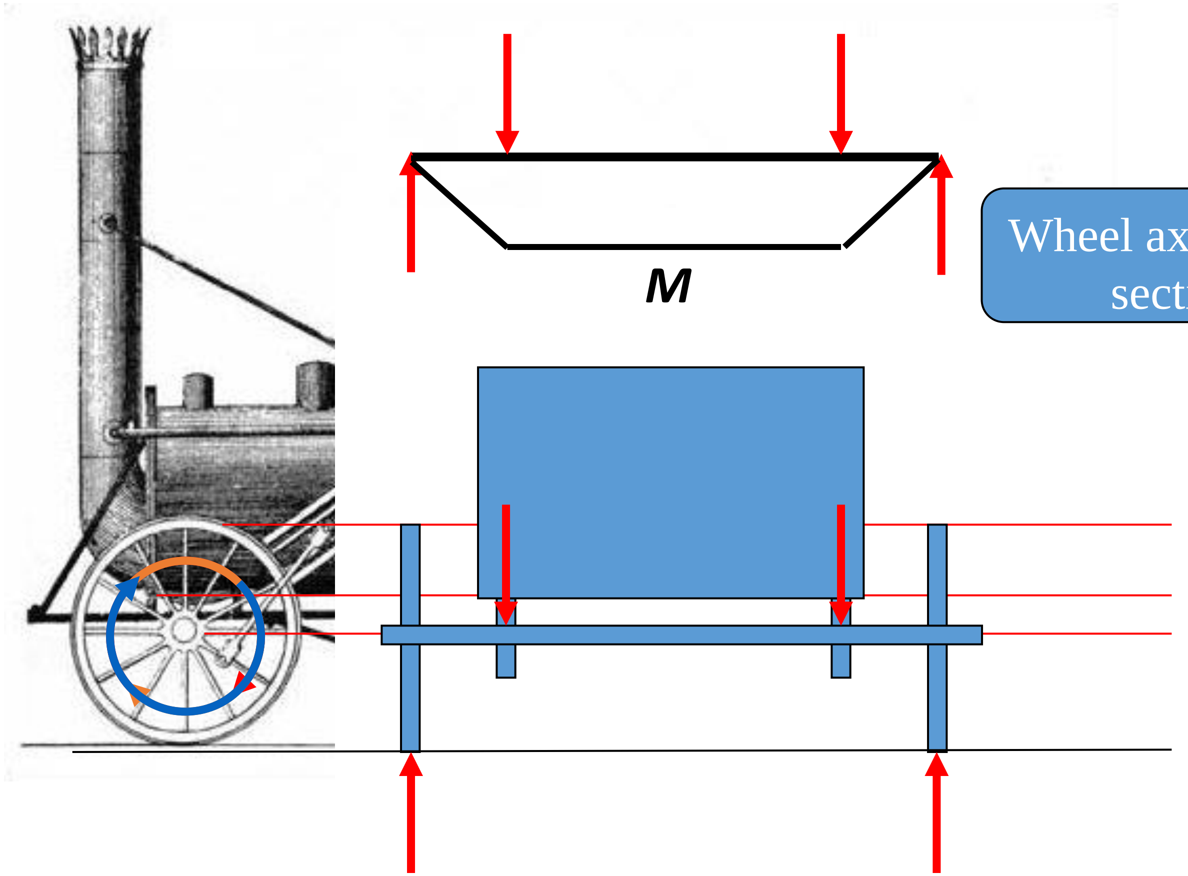
1. Fatigue life to the occurrence of a small detectable crack
2. Fatigue life to a certain percent decrease in load for constant strain cycling
3. Fatigue life to fracture

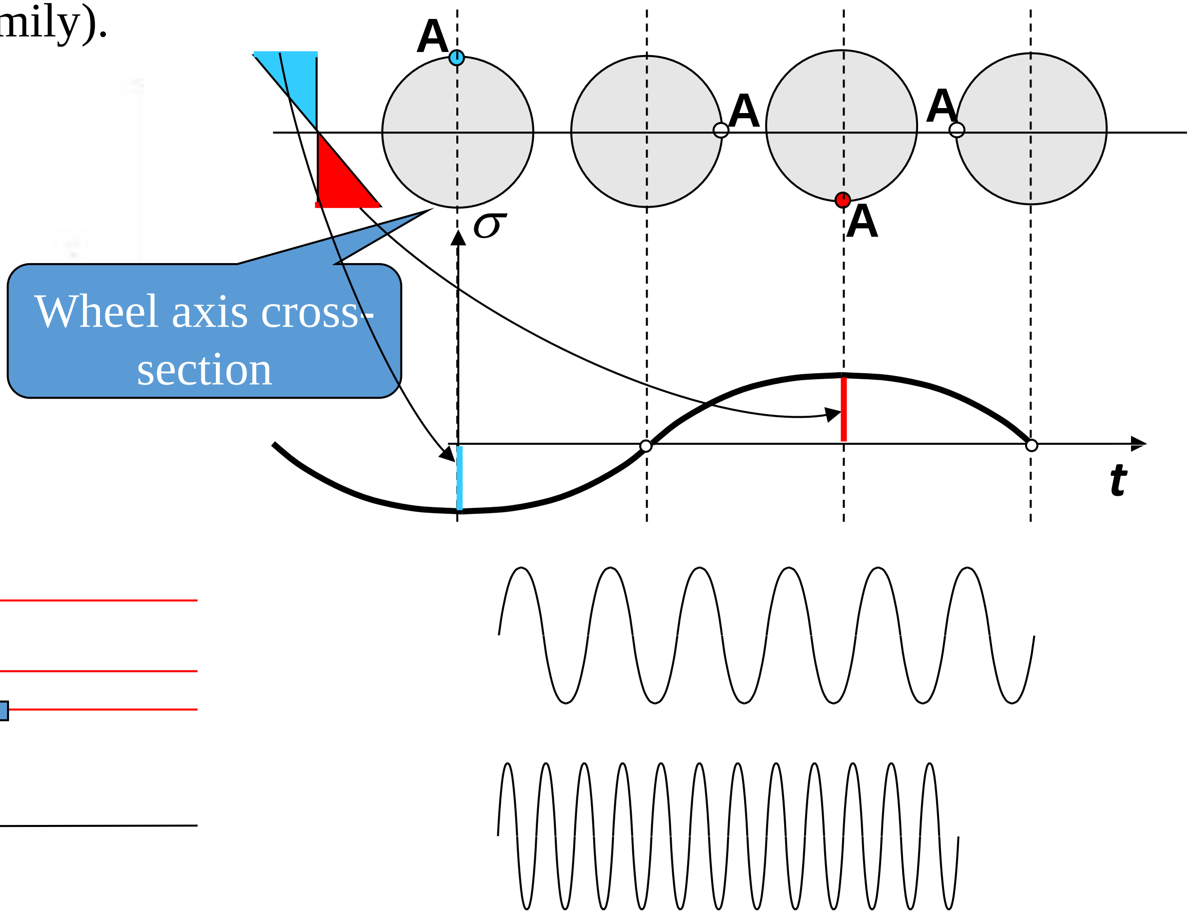


High cycle fatigue examples – broken axle

Fatigue appearance accelerated when steam railways were introduced in XIX century.

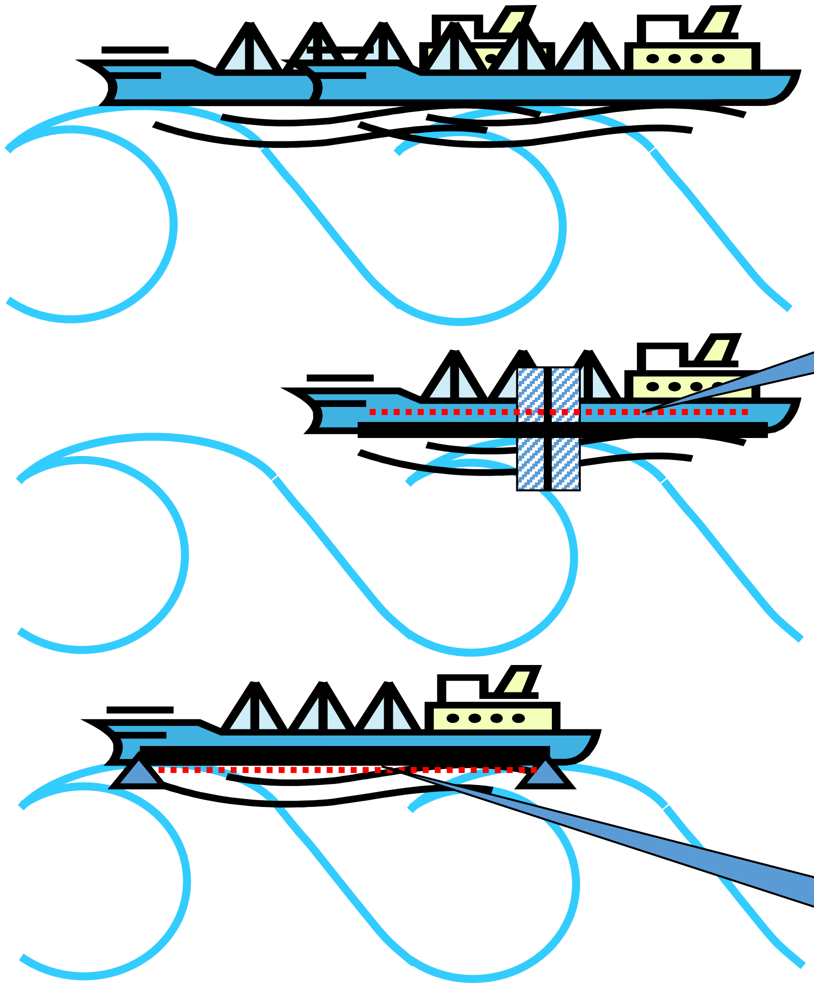
Versailles train crash (1842), at least 55 passengers were killed (including French rear admiral, botanist, cartographer and explorer Jules d'Urville and his family).





Fatigue examples – ships

A ship is working alternatively as double cantilever and simply supported beam

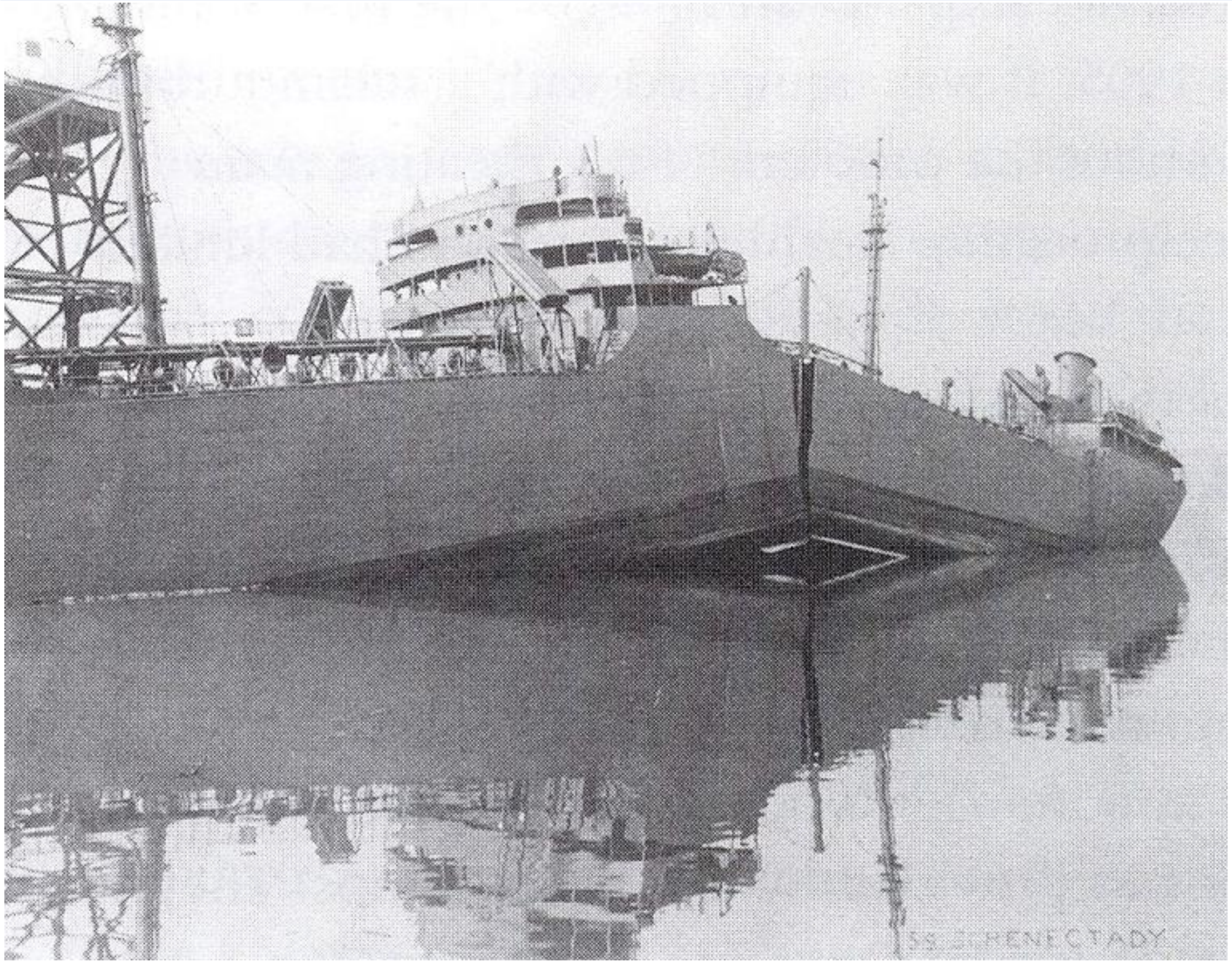


Deck under tension!
Keel under compression!

Until the middle of XX century the problem of ship cracking caused by fatigue remained unsolved.

Keel under tension!
Deck under compression!

Fatigue of ships cont.



During World War II, eighteen North-American shipyards built 3241 ships Liberty (2710) and Victory (531) of cargo class to replace losses caused by German submarines.

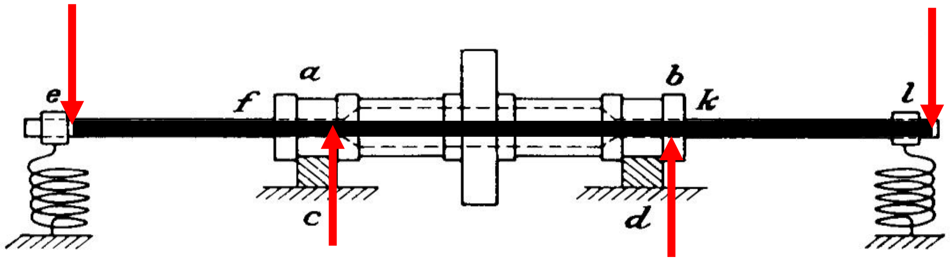
Tanker SS Schenectady, cracked in half when docked in the port on 16.01.1943, Portland, OR

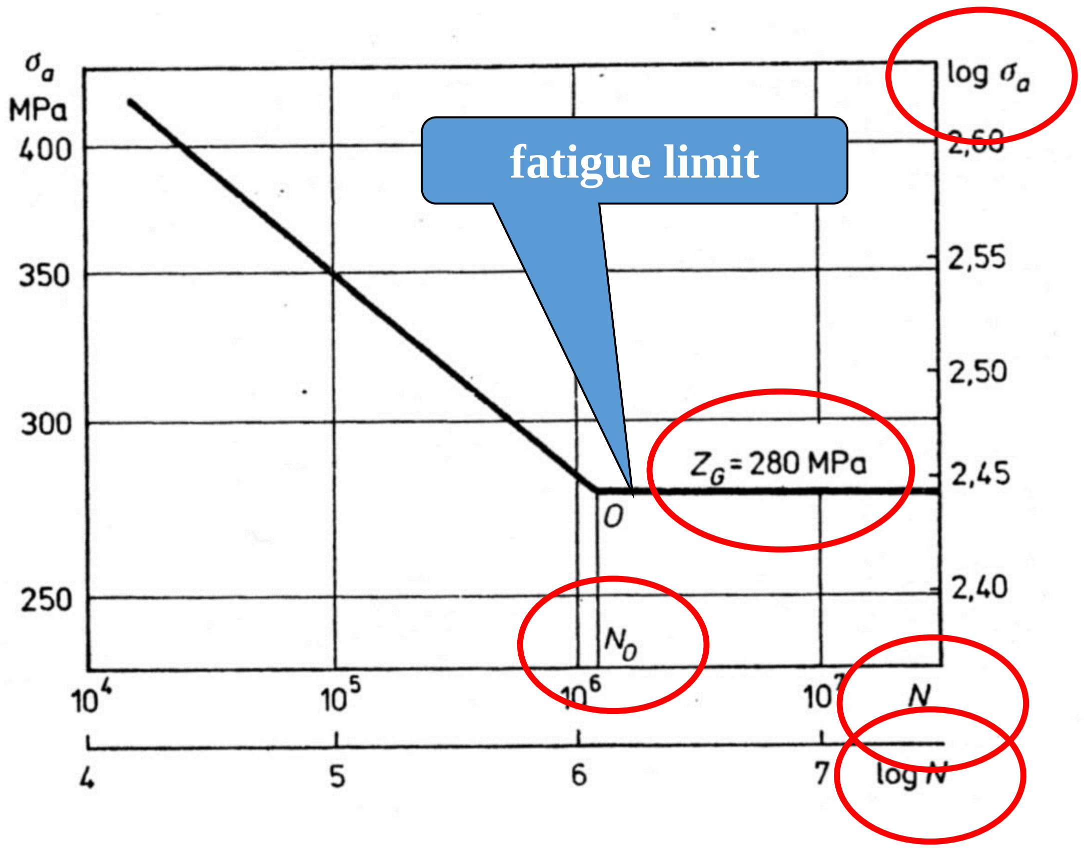
Fatigue examples – the Haviland Comet

Two de Havilland Comet passenger jets broke up in mid-air and crashed within a few months of each other in 1954.
The crash had been due to failure of the pressure cabin.
The failure was a result of metal fatigue caused by the repeated pressurization and de-pressurization of the aircraft cabin.
The sharp corners near the Comets' window openings acted as initiation sites for cracks.
The Comet's pressure cabin had been designed to a safety factor comfortably in excess of that required by British Civil Airworthiness Requirements (2.5 times the cabin pressure as opposed to the requirement of 1.33).
The accident caused a revision in the estimates of the safe loading strength requirements of airliner pressure cabins.
As a result, all future jet airliners would feature windows with rounded corners, greatly reducing the stress concentration.

Fatigue – Wöhler’s diagram

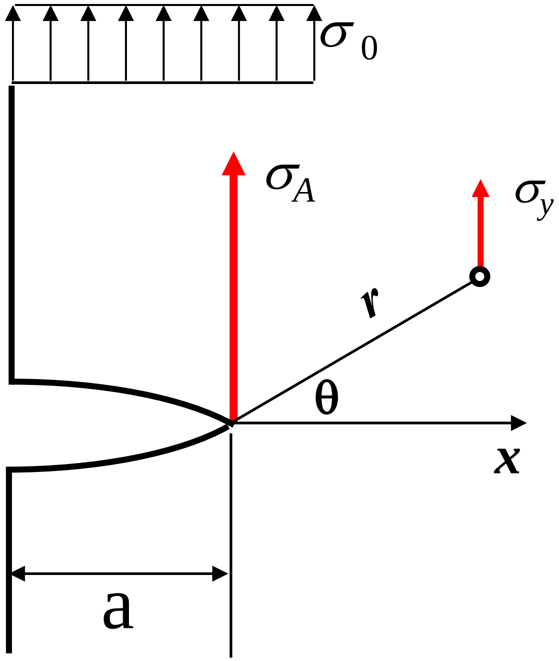
Wöhler’s stand for fatigue investigation





Wöhler’s diagram for high-cycle fatigue

Cracking



H.M.Westergaard, 1939, N.I.Muskhelischvili, 1943 – analyzed 2D stress field at the tip of a sharp slit

$$\sigma_y = \sigma_0 \sqrt{\frac{a}{2r}} \cos\frac{\theta}{2}\left(1 + \sin\frac{\theta}{2} + \sin\frac{3\theta}{2} + \ldots\right)$$

For $\theta \to 0$ $\sigma_y \to \sigma_A$

$$\sigma_A = \sigma_0 \sqrt{\frac{a}{2r}}$$

For $r \to 0$

singularity!

$$\sigma_A = \sigma_0 \sqrt{\frac{a\pi}{2r\pi}} = \frac{K}{\sqrt{2\pi r}}$$

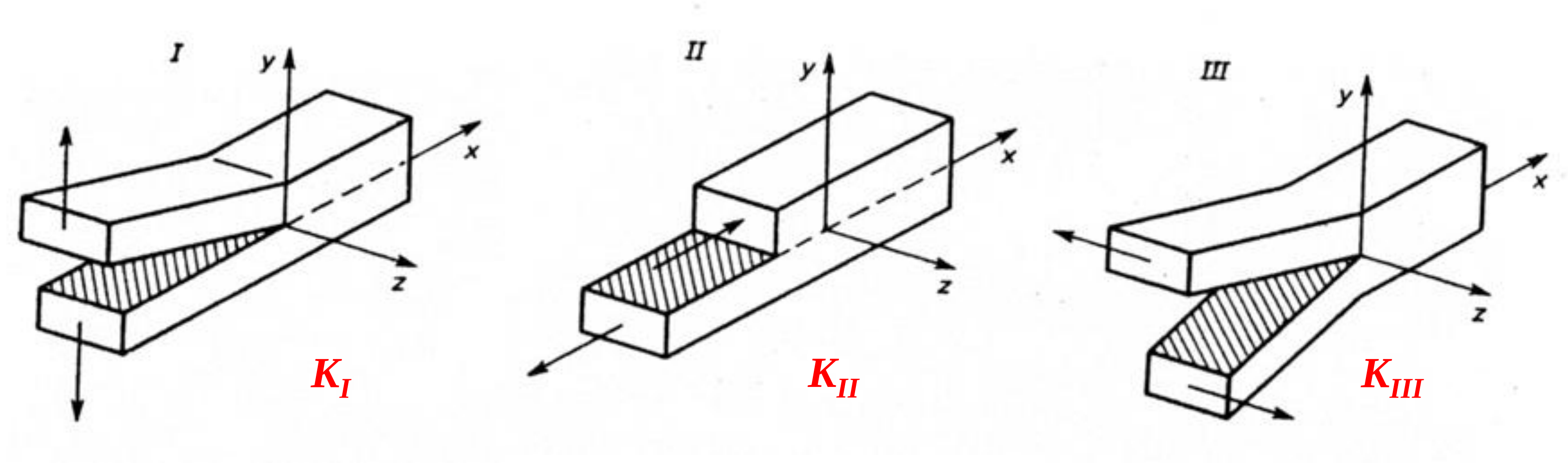
$$K = \sigma_0 \sqrt{\pi a}$$

Stress intensity factor

The significant defects (size scale) depend principally on the notch toughness of the material. Notch toughness is a measure of the ability of a material to absorb energy in the presence of a flaw.

Cracking – cont.

Stress intensity factors were calculated for different configuration of loading and specimen geometry (G. Sih)
Three typical cases (modes) can be distinguished:



Mode I – opening; the crack surfaces move directly apart (a tensile stress normal to the plane of the crack).
Mode II – sliding; the crack surfaces move normal to the crack tip and remain in the plane of the crack.
Mode III – tearing; the crack surfaces move parallel to the crack tip and remain in the plane of the crack.

Cracking – stress intensity factors

Design criteria are:

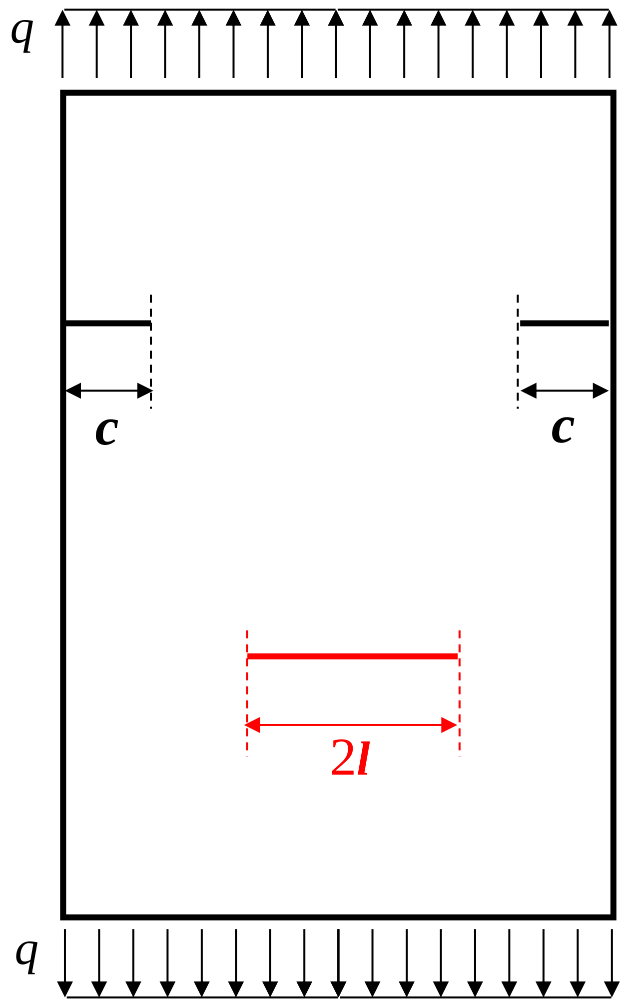
$$K_I < K_{Ic} \qquad K_{II} < K_{IIc} \qquad K_{III} < K_{IIIc}$$

where K_{Ic} , K_{IIc} , K_{IIIc} are critical values of corresponding stress factors, being determined experimentally.

Many materials can be made to behave in a ductile manner for a given set of conditions and in brittle manner for another set of conditions. One should speak of a material being in a brittle or ductile state. However, there is not always a clear demarcation between brittle and ductile states.
For the ductile state, it is possible to postulate failure criteria based on concepts of yielding flow beginning.
Brittle fracture problem are subdivided into three types:

1. brittle fracture of members free of cracks and flaws under static loading conditions
2. brittle fracture originating at cracks and flaws in member under static loading conditions
3. brittle fracture resulting from high cycle fatigue loading.

Fracture mechanics – an example



What is the length of a central slit we can introduce to the structure shown without the reduction of its load bearing capacity? (no interaction assumed)

$$2l \leq 2c \text{ or } 2l \geq 2c \ ?$$

For side crack $K_{I-side} = 1{,}12 q\sqrt{\pi c}$

For central crack $K_{I-center} = q\sqrt{\pi l}$

$$K_{I-center} \leq K_{I-side} \qquad q\sqrt{\pi l} \leq 1{,}12 q\sqrt{\pi c}$$

$$l \leq 1{,}12^2 c$$

$$l \leq 1{,}25c$$

If e.g. c = 2 cm

$2l \leq 5$ cm

Thank you for your attention!