Strength of Materials

13. Rheology

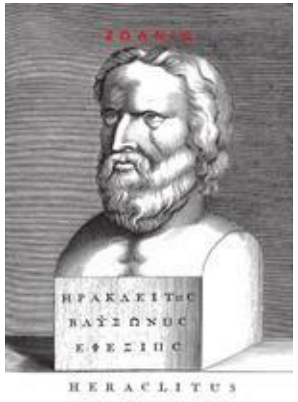
Introduction

Πάντα ῥεῖ καὶ οὐδὲν μένει

panta rhei
kai ouden menei

everything flows
and nothing remains still

Heraclitus of Ephesus (540-480 BC)
Ἡράκλειτος ὁ Ἐφέσιος
(*Herakleitos ho Ephesios*)



Rheology - time

Tempus fugit, aeternitas manet

t - time

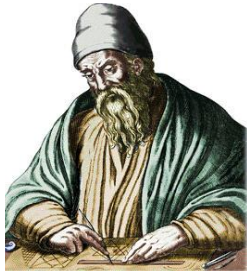
Hourglass is
the trademark of
Rheological Society

Pioneers…

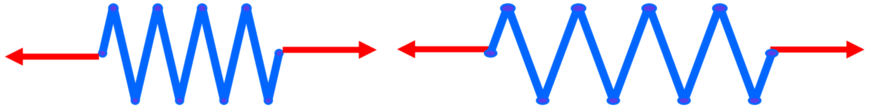
Solids

$p \neq 0$ $\varepsilon = 0$

Stiff body



Euclid, ~ 300 BC



$p \neq 0$ $\varepsilon \neq 0$

Elastic body

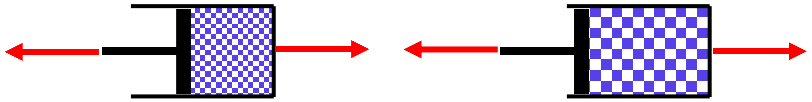
R. Hooke, 1635-1703

Fluids

$p \to 0$ $\varepsilon \to \infty$

Perfect fluid

B. Pascal, 1629-1662



$p \neq 0$ $\varepsilon \neq \text{const}$

Viscous fluid

I. Newton, 1643-1727

Time – definitions

Time – a new independent variable for description of materials' behavior and/or processes course (by load, temperature, humidity, corrosion, etc.)

Three aspects:

- creep: slow increase of strain with stress constant in time or change of the stress under constant strain
- ageing: change of material mechanical properties with time (concrete)
- memory: materials remember processes of the past

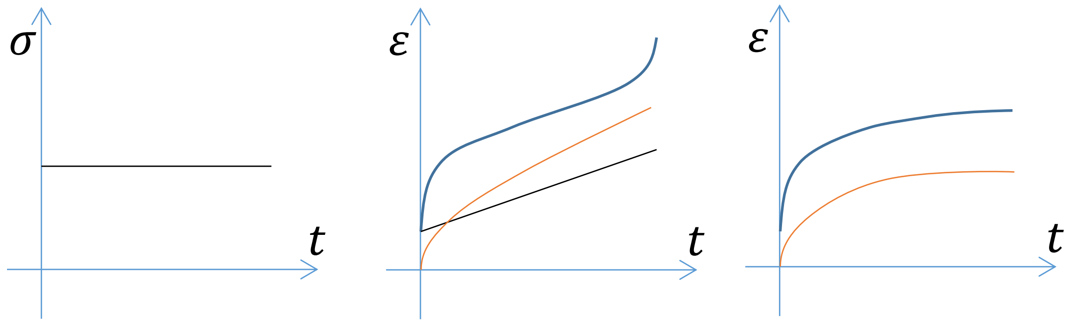
There are the viscous materials, visco-elastic materials, and visco-plastic materials (sometimes known as visco-elasto-plastic materials and visco-elasto/plastic materials) /the slash means a distinct elastic limit/

There are two aspects of creep in structures:

- flow – a change (usually very slow) od strain under constant stress
- relaxation – change of stress under constant strain

There are two type of the structural materials:

- materials of fluid type (unlimited flow)
- materials of solid type (limited flow)

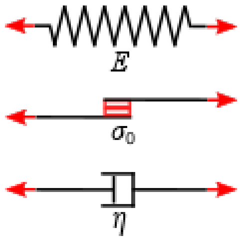


Structural models

Three type of elements:

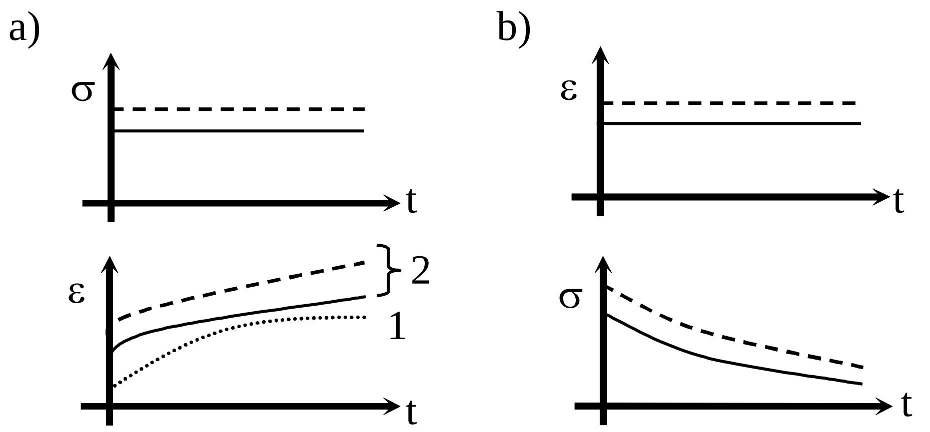
- Hooke's element (elastic): $\varepsilon = \sigma / E$
- de Saint-Venant element (plastic): $\varepsilon = \begin{cases} 0 \text{ for } \sigma < R_e \\ \neq 0 \text{ for } \sigma \geq R_e \end{cases}$
- Newton element (viscous): $\dot{\varepsilon} = \sigma / \eta$

$$\eta \left[\frac{N}{m^2} s\right]$$



Two basic tests:

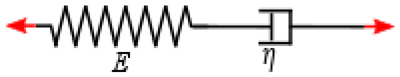
- creep, $\sigma(t) = \sigma_0 = \text{const}$
- relaxation, $\varepsilon(t) = \varepsilon_0 = \text{const}$



a) creep: 1 – limited, 2 – unlimited
b) relaxation

Maxwell model

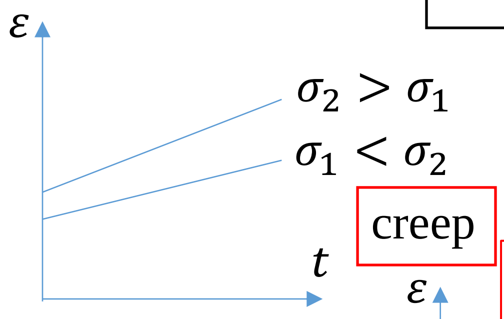
connection in series of the Hooke and Newton models



$$\varepsilon = \varepsilon_{el} + \varepsilon_v \qquad \varepsilon_{el} = \frac{\sigma}{E} \qquad \dot{\varepsilon}_v = \frac{\sigma}{\eta} \qquad \boxed{\dot{\varepsilon} = \frac{\dot{\sigma}}{E} + \frac{\sigma}{\eta}}$$

rheological constitutive equation
(rheological state equation)

creep test: $\sigma = \sigma_0 \quad \dot{\varepsilon} = \frac{\sigma_0}{\eta} \rightarrow \varepsilon(t) = \frac{\sigma_0}{\eta} t + C$ initial condition: $\varepsilon(t=0) = \frac{\sigma_0}{E} \rightarrow C = \frac{\sigma_0}{E} \rightarrow \boxed{\varepsilon = \frac{\sigma_0}{E} + \frac{\sigma_0}{\eta} t}$



relaxation test: $\varepsilon = \varepsilon_0$
we seek solution to the rheological constitutive equation by:

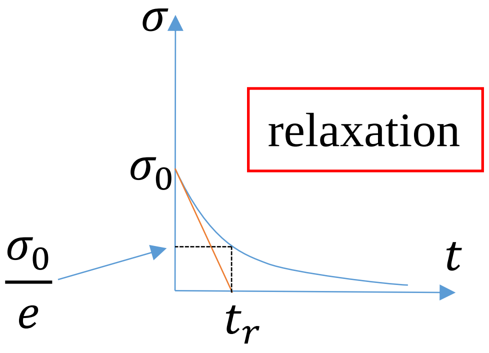
$$\sigma(t) = Ce^{rt} \rightarrow \dot{\sigma} = rCe^{rt} \rightarrow 0 = \frac{rCe^{rt}}{E} + \frac{Ce^{rt}}{\eta} \rightarrow \eta r + Er = 0$$

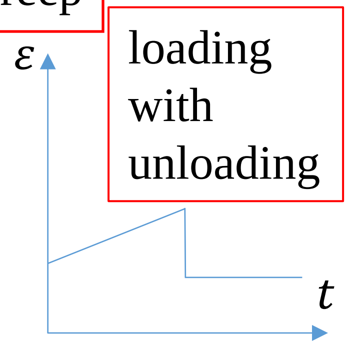
$r = -\frac{E}{\eta}$, initial condition: $\sigma(t=0) = \sigma_0 \rightarrow C = \sigma_0 \rightarrow \sigma = \sigma_0 e^{-\frac{E}{\eta}t}$

$\frac{\eta}{E} \stackrel{\text{def}}{=} t_r$ [s] relaxation time

$$\boxed{\sigma = \sigma_0 e^{-\frac{t}{t_r}}}$$

for $t = t_r \rightarrow \sigma = 0.3679\sigma_0$

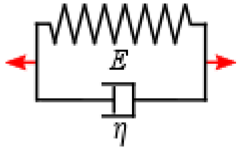




the Maxwell material relaxes totally, the relaxation rate is characterized by the relaxation time (the material constant)

Kelvin model

parallel connection of the Hooke and Newton models



$\sigma=\sigma_{el}+\sigma_v \qquad \sigma_{el}=E\varepsilon \qquad \sigma_v=\eta\dot{\varepsilon} \qquad \boxed{\sigma=E\varepsilon+\eta\dot{\varepsilon}}$

creep test: $\sigma=\sigma_0$

$\dot{x}(t)+p(t)x(t)=f(t)\to x(t)=e^{-\int p(t)dt}\left\{\int f(t)e^{\int p(\tau)d\tau}dt+C\right\}$, so:

$$\dot{\varepsilon}+\frac{E}{\eta}\varepsilon=\frac{\sigma_0}{\eta}\to\ \varepsilon=e^{-\frac{E}{\eta}t}\frac{\sigma_0}{\eta}\int e^{\frac{E}{\eta}\tau}dt=e^{-\frac{E}{\eta}t}\left\{\frac{\sigma_0}{\eta}\cdot\frac{\eta}{E}\cdot e^{\frac{E}{\eta}t}+C\right\}=\frac{\sigma_0}{E}+Ce^{-\frac{E}{\eta}t}$$

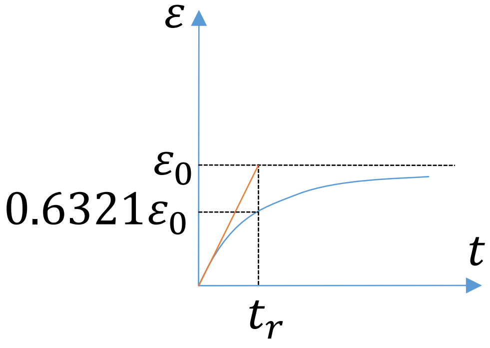
the initial condition: $\varepsilon(t=0)=0\to C=-\frac{\sigma_0}{E}\ \to\ \varepsilon=\frac{\sigma_0}{E}\left(1-e^{-\frac{E}{\eta}t}\right)$

$t_d\overset{\text{def}}{=}\frac{\eta}{E}$ [s] retardation time

$$\boxed{\varepsilon=\frac{\sigma_0}{E}\left(1-e^{-\frac{t}{t_d}}\right)}$$

nonsteady (nonlinear) bounded creep

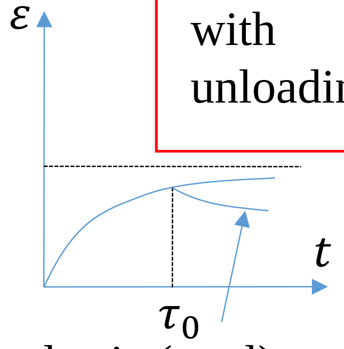
there is no relaxation



creep

$$\varepsilon(t)=\frac{\sigma_0}{E}\left[e^{-\frac{t-\tau_0}{t_d}}-e^{-\frac{t}{t_d}}\right]$$

loading with unloading



inelastic (total) recovery

Operator method for constitutive equation

Jan Stefan Mikusiński (1913-1987)
Mikusiński's operator

$$D \stackrel{\text{def}}{=} \frac{d}{dt}$$

can be treated as an algebraic quantity
for ex. Kelvin model: $\sigma(t) = E\varepsilon(t) + \eta\dot{\varepsilon}(t) = E\varepsilon(t) + \eta D\varepsilon(t) = (E + D\eta)\varepsilon(t)$
module of model:

$$M(D) \stackrel{\text{def}}{=} \frac{\sigma(t)}{\varepsilon(t)}$$

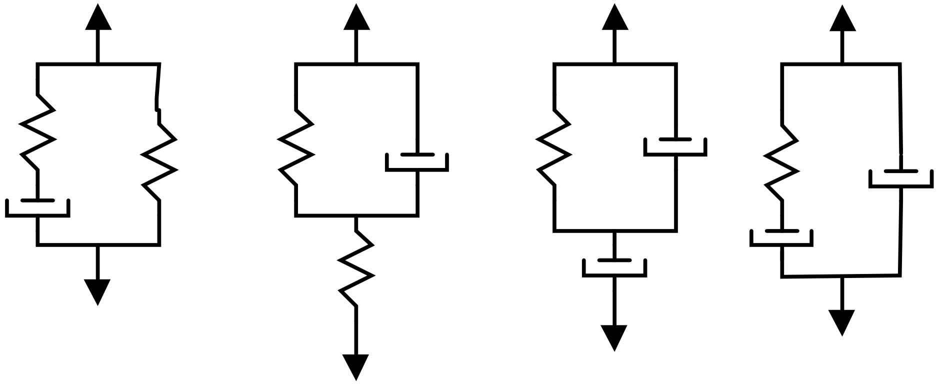
so, for Kelvin model the module is: $M(D) = E + D\eta$
for connection in series of *n* models

$$\varepsilon(t) = \sum_{i=1}^{n} \varepsilon_i(t), \qquad \sigma_i(t) = \sigma(t) = \text{idem} \rightarrow \frac{1}{M(D)} = \frac{\varepsilon(t)}{\sigma(t)} = \frac{\sum_{i=1}^{n} \varepsilon_i(t)}{\sigma_i(t)} = \sum_{i=1}^{n} \frac{\varepsilon_i(t)}{\sigma_i(t)} = \sum_{i=1}^{n} \frac{1}{M_i}$$

for parallel connection of *n* models

$$\sigma(t) = \sum_{i=1}^{n} \sigma_i(t), \qquad \varepsilon_i(t) = \varepsilon(t) = \text{idem} \rightarrow \sum_{i=1}^{n} M_i = M(D)$$

Standard models



a) M||H, b) K-H, c) K-N, d) M||N

An example: Kelvin + Newton $(E, \eta_1 + \eta_2)$

$$\frac{1}{M(D)} = \frac{1}{M_K} + \frac{1}{M_N} = \frac{1}{E + D\eta_1} + \frac{1}{D\eta_2} = \frac{D\eta_2 + E + D\eta_1}{D^2\eta_1\eta_2 + DE\eta_2}$$

$$\frac{\sigma(t)}{\varepsilon(t)} = \frac{D^2\eta_1\eta_2 + DE\eta_2}{D\eta_2 + E + D\eta_1} \rightarrow D^2\eta_1\eta_2\varepsilon(t) + DE\eta_2\varepsilon(t) = D\eta_2\sigma(t) + D\eta_1\sigma(t) + E\sigma(t)$$

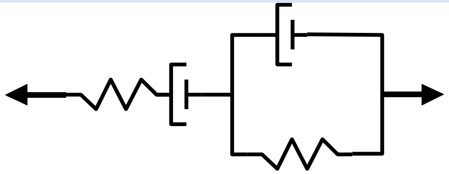
so, finally

$$E\sigma(t) + (\eta_1 + \eta_2)\dot{\sigma}(t) = E\eta_2\dot{\varepsilon}(t) + \eta_1\eta_2\ddot{\varepsilon}(t)$$

The equation order is equal to the number of separate viscous elements (two viscous elements, connected in series or parallel, should be treated as one viscous element)

Burgers' model

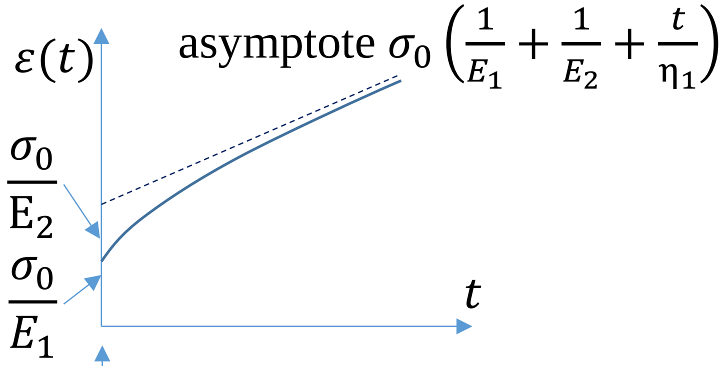
M (1) and K (2) in series:



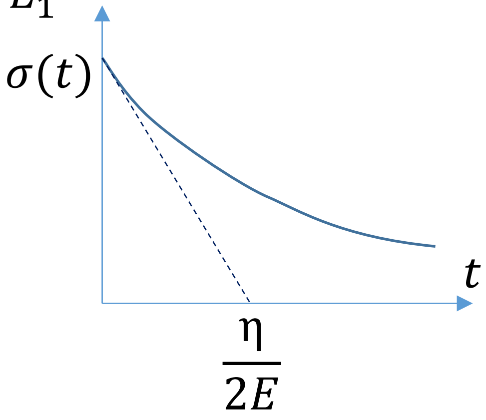
rheological equation of the state: $\sigma(t) + \left(\frac{\eta_1}{E_1} + \frac{\eta_1}{E_2} + \frac{\eta_2}{E_2}\right)\dot{\sigma}(t) + \frac{\eta_1\eta_2}{E_{!}E_2}\ddot{\sigma}(t) = \eta_1\dot{\varepsilon}(t) + \frac{\eta_1\eta_2}{E_2}\ddot{\varepsilon}(t)$

creep test:

$$\varepsilon(t) = \sigma_0\left\{\frac{1}{E_1} + \frac{t}{\eta_1} + \frac{1}{E_2}\left[1 - \exp\left(-\frac{E_2}{\eta_2}t\right)\right]\right\}$$



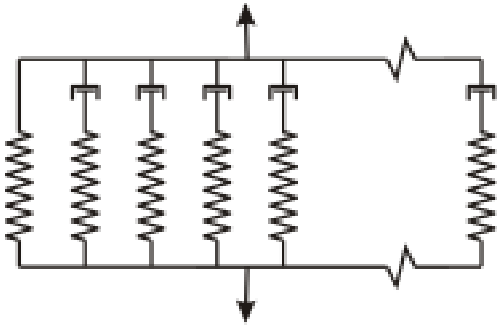
relaxation test (for $E_1 = E_2 = E$ and $\eta_1 = \eta_2 = \eta$)



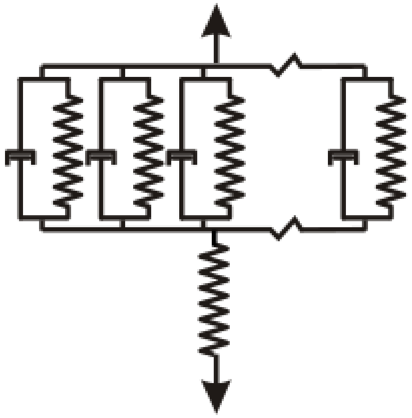
General form of state equation for simple models

$$A_0\varepsilon(t) + A_1\dot{\varepsilon}(t) + A_2\ddot{\varepsilon}(t) + \cdots + A_n\varepsilon^{(n)}(t) = B_0\sigma(t) + B_1\dot{\sigma}(t) + B_2\ddot{\sigma}(t) + \cdots + B_m\sigma^{(m)}(t)$$

generalized Maxwell model



generalized Kelvin model

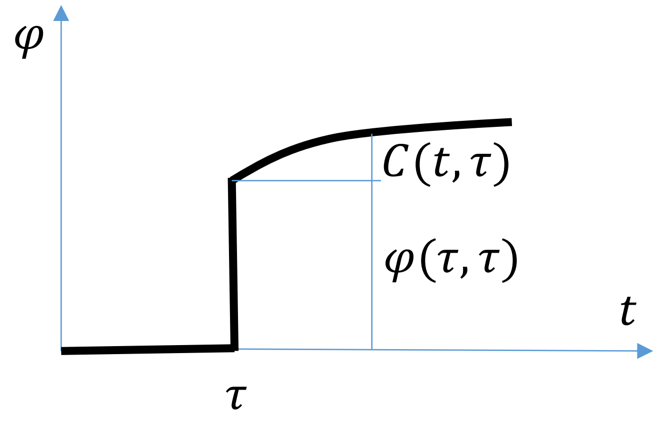


Solution to the problems with the structural models, demanding differential equations of the second or higher order, can be very complicated. This is more advisable to construct the rheological equations of state in form of integral equations (the hereditary theories).

Linear hereditary theory

creep function, $\varphi(t)$:

$$\varphi(t) \stackrel{\text{def}}{=} \frac{\varepsilon(t)}{\sigma_0}$$



observation time, t
age (loading time), τ
The hereditary theory:
creep function, $\varphi(t,\tau)$
creep measure, $C(t,\tau) = \varphi(t,\tau) - \varphi(\tau,\tau)$
The invariant hereditary theory (loading duration counts only):

$$C(t,\tau) = C(t-\tau)$$

rheological equation of state

$$\varepsilon(t) = \frac{1}{E}\left[\sigma(t) + \int_{\tau_0}^{t} \sigma(\tau)K(t-\tau)d\tau\right]$$

$K(t-\tau)$ – the kernel of the Volterra integral equation of the second kind; K is an influence function of the stimulus $\sigma(\tau)d\tau$ on the strain; this describes material with fading memory
Boltzmann superposition principle is valid: $\varepsilon(\sigma_1+\sigma_2) = \varepsilon(\sigma_1) + \varepsilon(\sigma_2)$

Linear hereditary theory – cont.

The non-invariant hereditary theory (used for concrete)

$$\varepsilon(t) = \frac{1}{E(t)}\left[\sigma(t) + \int_{\tau_0}^{t} \sigma(\tau)K(t,\tau)d\tau\right]$$

If the kernel is degenerated:

- $f(t)g(\tau)$, or
- $f(t-\tau)$

the Volterra equation come down to the differential equations with variable or constant coefficients, respectively.
Both hereditary theories describe the previous history of loading or deformation. The material memory is fading: the recent events are better remembered than the forepast events. The influence of older loading on actual state of strain is weaker than the influence of the recent process.

Phenomenological theories of creep

the creep strains:

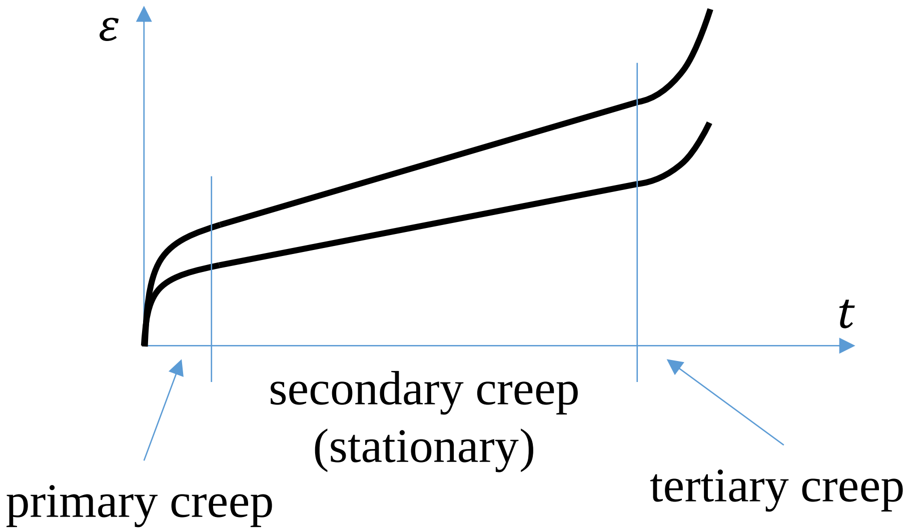
$$\varepsilon_c = f_1(\sigma) f_2(t) f_3(T)$$

where T - absolute temperature

the most used propositions are:

- for $f_1(\sigma)$
  - $B\sigma^n$ (Norton 1929, the most prevalent and verified)
  - $A\left\{\sinh\left[\left(\frac{\sigma}{\sigma_0}\right)\right]\right\}^n$ (Garofalo 1965)
- for $f_2(t)$
  - $\{1 + at^{1/3}\}\exp(bt)$ (Andrade 1910)
  - ct^m (Bailey 1935)
- for $f_3(T)$
  - $\exp\left(-\frac{Q}{RT}\right)^p$ (Dorn 1955)

typical creep curves for metallic alloys:



Summary of creep

There are two fundamental characteristics of rheological processes:

- Its dependence on the history of loading
- Energy dissipation - causing irreversibility

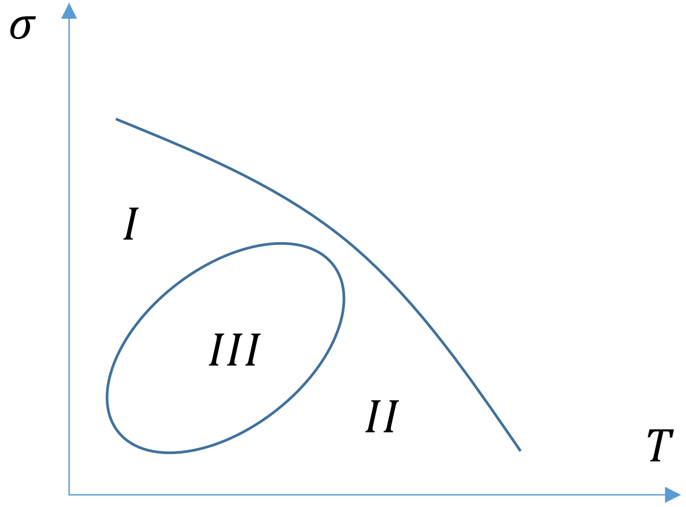
Macroscopically observable effects are due to material microstructure changes (see material science and Ashby maps).

These changes can lead not only to irreversible deformation and stress relaxation but to the formation and growth of microstructural defects.

Following this deterioration process a structure can be fatally damaged at arbitrary level of loading or deformation – after a sufficiently long period of loading time.

This is, however, a subject of another important branch of solid mechanics – mechanics of damage and failure.

Damage under creep conditions



I – ductile damage (high stress, low temperature, slips prevail)
II – brittle damage (low stress, high temperature, cracks prevail)
III – mixt damage (various mechanisms)

Ductile and brittle damage

Hoff 1953

material: $\dot{\varepsilon} = B\sigma^n$

$l \to \infty, A \to 0$

Hoff's damage time: $t_H = \dfrac{1}{nB\sigma_0^n}$

Kachanov 1958 continuity parameter: $\psi \overset{\text{def}}{=} \dfrac{A}{A_0}$ an evolution equation $\dfrac{d\psi}{dt} = -C\left(\dfrac{\sigma_0}{\psi}\right)^m$

Kachanov's damage time: $t_K = \dfrac{1}{(m+1)C\sigma_0^m}$

mixt damage (ductile-brittle damage) $\dfrac{\mathrm{t_m}}{\mathrm{t_H}} = 1 - \left[1 - \dfrac{t_K}{t_H}\dfrac{n-m}{n}\right]^{\frac{n}{n-m}}$

Rheology in civil engineering

loss of prestress

the essence of prestressing concrete is that once the initial compression has been applied, the resulting material has the characteristics of high-strength concrete when subject to any subsequent compression forces, and of ductile high-strength steel when subject to tension forces; this can result in improved structural capacity and/or serviceability compared to conventionally reinforced concrete in many situations

support's settlement

defective foundation, bad soil conditions; usually the settlement process takes a lot of time (months or years)

concrete creep resulting in the stress redistribution

Thank you for your attention!