Theory of Plasticity

Introductory remarks

Plasticity – inelasticity with perfect material memory

Course details: 12 lecture hours & 8 designing hours

The scope of the plasticity theory

- plastic behavior of metals and ceramic materials
- constitutive equations for plastic materials within thermodynamic internal variables theory
- technological problems of plastic processing (forging, rolling, stamping, extrusion, pull broaching and others),
- limit bearing capacity of the bar structures (mostly in bending), plates and in soil mechanics,
- dynamic problems,
- large-deformation plasticity

Some themes are out of scope of our considerations.

References

Classical books:

- Hill R., *The mathematical theory of plasticity*, Oxford, 1950
- Kachanov L. M., *Fundamentals of theory of plasticity*, Mir ed., Moscow, French ed. 1975,
- Olszak W., Perzyna P., Sawczuk A., *Teoria plastyczności*, PWN, Warszawa 1975
- Życzkowski M., *Combined loading in the theory of plasticity*, PWN, Warszawa 1981
- Chen W.-F., Han D. J., *Plasticity for structural engineers*, Springer, NY-Berlin-Heidelberg-London-Paris-Tokyo, 1988
- Chen W.-F., *Limit analysis and soil plasticity*, Elsevier, Amsterdam-Oxford-NY 1975

Textbooks:

- Krzyś W., Życzkowski M., *Sprężystość i plastyczność. Wybór zadań i przykładów*, PWN, Warszawa 1962
- Skrzypek J., *Teoria plastyczności i pełzania*, CUT's printed lectures, 1975
- Gabryszewski Z., *Teoria sprężystości i plastyczności*, Wrocław Techn. Univ. ed. 2001

Modern "more demanding" books

- Perzyna P., T*ermodynamika materiałów niesprężystych*, PWN, Warszawa 1978

- Rymarz Cz., *Mechanika ośrodków ciągłych*, PWN, Warszawa 1993
- Skrzypek J., Hetnarski R., *Plasticity and creep: Theory, examples and problems*, CRC Press, Florida 1993

Modern books available in Internet:
- Chakrabarty J., *Theory of plasticity*, Elsevier Butterworth-Heinemann, 2006
- Chakrabarty J., *Applied plasticity*, Springer, NY-Dordrecht-Heidelberg-London 2010
- Lubliner J., *Plasticity theory*, Pearson Edu., Berkeley 2000

Some statements from mechanics of continua

Volumetric and deviatory strain can be split into:

$$T_\varepsilon = A_\varepsilon + D_\varepsilon$$

In the same way, we decompose:

$$T_\sigma = A_\sigma + D_\sigma$$

and, on the octahedral plane:

$$\mathbf{n} = \frac{1}{\sqrt{3}}(\pm e_1, \pm e_2, \pm e_3)$$

we have:

$$\sigma(\mathbf{n}) = \sigma_m, \qquad |\tau(\mathbf{n})|^2 = \tau_{oct}^2 = \tfrac{2}{3}J_2$$

Boundary value problem

The set of differential equations with the boundary conditions with the displacement and/or surface traction prescribed is called a boundary value problem.

Admissible fields:
- kinematically admissible displacement field,
- statically admissible stress field.

Virtual displacements – the difference between two kinematically admissible displacement fields, δu_i.

Principle of virtual work (principle of virtual displacements):

$$\int_R (\sigma_{ij,j} + \rho b_i)\delta u_i dV - \int_{\partial R_t} (n_j\sigma_{ij} - t_i^\alpha)\delta u_i dS = 0$$

may be interpreted as an application of the method of weighted residuals. When a certain stress field is not exactly statically admissible, we have:

$$\sigma_{ij,j} + \rho b_i + \rho\Delta b_i = 0 \text{ in } R, \qquad n_j\sigma_{ij} = t_i^\alpha + \Delta t_i^\alpha \text{ on } \partial R_t$$

$\Delta\mathbf{b}$ and Δt^α being the residuals of the body force and applied surface traction, respectively. We can try to make them vanish in some average sense, multiplying them with a vector-valued weighting function, **w**, such that:

$$\int_R \rho\Delta\mathbf{b}\cdot\mathbf{w} dV + \int_{\partial R_t} \Delta t_i^\alpha w_i ds = 0$$

for every **w** belonging to *W*.

In the light of the method of weighted residuals, the principle of virtual work may be represented by the equation:

$$\int_R (\sigma_{ij,j} + \rho b_i) w_i dV - \int_{\partial R_t} (n_j \sigma_{ij} - t_i^\alpha) w_i dS = 0$$

Virtual stress field – the difference between two statically admissible stress fields, $\delta\boldsymbol{\sigma}$.
Principle of virtual forces (principle of complementary virtual work):

$$\int_R \left[\varepsilon_{ij} - \tfrac{1}{2}(u_{i,j} + u_{j,i})\right] \delta\sigma_{ij} dV + \int_{\partial R_u} (u_i - u_i^\alpha) n_j \delta\sigma_{ij} dS = 0$$

Inelasticity

Time as a new independent variable:

- rate sensitivity, deformation produced by slow stressing is usually greater than that produced by rapid stressing
- flow, (usually slow) change of displacements in time for constant stresses and/or slow decrease of the stress at a fixed strain
- ageing, material characteristics change in time
- material memory, actual material behavior depends on previous material's history

Surpassing the elastic limit, material yields. The elastic range forms a region in the space of the stress components, usually called the elastic region and its boundary is called the yield surface.
For infinitesimal deformation, it is almost universally assumed that the strain tensor can be decomposed additively into an elastic strain and an inelastic strain:

$$\varepsilon_{ij} = \varepsilon_{ij}^e + \varepsilon_{ij}^i$$

Internal variables
Scalar or second-rank tensor values, ξ:

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\boldsymbol{\sigma}, T, \xi)$$

Additional constitutive equations are required (so-called the equations of evolution or rate equations) for the internal variables:

$$\dot{\xi}_\alpha = g_\alpha(\boldsymbol{\sigma}, T, \xi)$$

Linear viscoelasticity

Standard solid model:

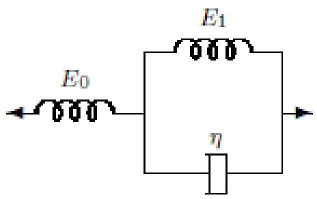

Fig. 1.1 Standard solid model

$$\sigma = E_0 \varepsilon^e, \quad \sigma = E_1 \varepsilon^i + \eta \dot{\varepsilon}^i$$

Total strain:

$$\varepsilon = \frac{\sigma}{E_0} + \varepsilon^i$$

$$\dot{\varepsilon}^i = \frac{1}{\eta}\sigma - \frac{E_1}{\eta}\varepsilon^i$$

The inelastic strain is an internal variable, the last equation being its rate equation. For given stress as a time function the evolution equation of inelastic strain can be solved:

$$\varepsilon^i(t) = \frac{1}{\eta}\int_{-\infty}^{t} e^{-(t-\tau)/t_d}\sigma(\tau)d\tau$$

where the reference time (at which $\varepsilon^i = 0$ is chosen as $-\infty$ for convenience, and $t_d = \eta/E_1$ is a material property having the dimension of time. In particular, if:

$$\sigma(t) = \begin{cases} 0 \text{ for t<0} \\ \sigma_0 \text{ for } t > 0 \end{cases}$$

then we get a form of creep known as delayed elasticity:

$$\varepsilon^i(t) = \frac{1}{E_1}\left(1 - e^{t/t_d}\right)\sigma_0$$

For $E_1 = 0$ we get the Maxwell model with infinite retardation time, so the exponential factor inside the integral becomes unity, and the creep solution displays steady creep:

$$\varepsilon^i(t) = \frac{\sigma_0}{\eta}t$$

Generalized Kelvin model

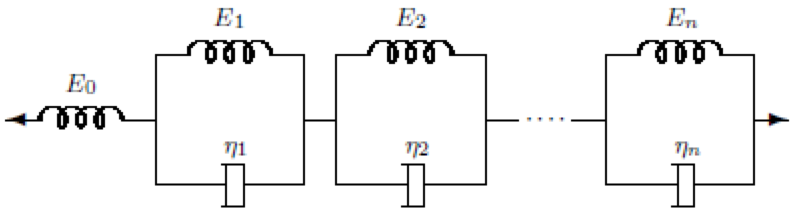

Fig. 1.2 Generalized Kelvin model

Every dashpot displacement constitutes an internal variable:

$$\varepsilon^i = \sum_{\alpha=1}^{n}\xi_\alpha$$

By analogy, the evolution equations are:

$$\dot{\xi}_\alpha = \frac{\sigma}{\eta_\alpha} - \frac{\xi_\alpha}{t_{d\alpha}}$$

and can be integrated explicitly:

$$\xi_\alpha(t) = \int_{-\infty}^{t}\frac{1}{\eta_\alpha}e^{-\frac{t-\tau}{t_d}}\sigma(\tau)d\tau$$

The total strain is:

$$\varepsilon(t) = \frac{1}{E_0}\sigma(t) + \int_{-\infty}^{t}\left(\sum_{\alpha=1}^{n}\frac{1}{\eta_\alpha}e^{-\frac{t-\tau}{t_{d\alpha}}}\right)\sigma(\tau)d\tau$$

Introducing the uniaxial creep function $J(t)$:

$$J(t) = \frac{1}{E_0} + \sum_{\alpha=1}^{n}\frac{1}{E_\alpha}\left(1 - e^{-t/t_{d\alpha}}\right)$$

the strain can be expressed (by integration by parts) as:

$$\varepsilon(t) = \int_{-\infty}^{t} J(t-\tau)\frac{d\sigma}{d\tau}d\tau$$

If

$$\sigma(\tau) = \begin{cases} 0 \text{ for } \tau < 0 \\ \sigma \text{ for } \tau > 0 \end{cases}$$

then

$$\varepsilon(t) = \sigma J(t)$$

therefore, the creep function can be determined experimentally from a single creep test.
Similarly, the same can be done for a relaxation test:

$$\varepsilon(\tau) = \begin{cases} 0 \text{ for } \tau < 0 \\ \varepsilon \text{ for } \tau > 0 \end{cases}$$

the stress is

$$\sigma(t) = \varepsilon R(t)$$

where $R(t)$ is the uniaxial relaxation function, and:

$$\sigma(t) = R(t-\tau)\frac{d\varepsilon}{d\tau}d\tau$$

An explicit form of the relaxation function in terms of internal variables can be obtained from the generalized Maxwell model, in Fig.

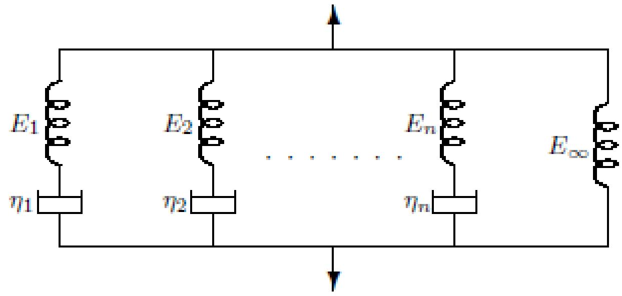

Fig. 1.3 Generalized Maxwell model

Internal variables: General theory

In a local equilibrium state the internal variables remain constant. Nonequilibrium states are an essential feature of rate-dependent inelastic continua. The states evolve in time by means of irreversible processes. In the thermodynamics of irreversible processes the temperature and the entropy are defined at a nonequilibrium state. The second law of thermodynamics can be expressed by the local Clausius-Duhem inequality. The inequality is obeyed if and only if material obeys the dissipation inequality (called also Kelvin inequality):

$$D = \sum_{\alpha} p_\alpha \dot{\xi}_\alpha \geq 0$$

where the thermodynamic force conjugate to the internal variable ξ_α is:

$$p_\alpha = -\rho \frac{\partial \psi}{\partial \xi_\alpha}$$

and ψ is the Helmholtz free energy density. Note that only the sum should be nonnegative and every term of dissipation inequality may be negative without violating the second law of thermodynamics.

Two types of internal variables:
- physical, describing aspects of local physic-chemical structure (extent of the reaction, relative density of two phases, etc.)
- phenomenological, being some mathematical constructs where the form of the functional dependence is assumed a priori (inelastic strain)

Flow law and flow potential

A flow law is an evolution equation for inelastic strain. It is assumed that the rate equations are (generalized normality for e generalized potential Ω):

$$\dot{\xi}_\alpha = \frac{\partial \Omega}{\partial p_\alpha}$$

The thermodynamics forces p_α can be obtained as function of stress by means of the complementary free-energy density (free-enthalpy density or Gibbs function):

$$\chi = \rho^{-1}\sigma_{ij}\varepsilon_{ij} - \psi$$

(ψ – Helmholtz free-energy density). It can be easily shown that:

$$p_\alpha = \rho \frac{\partial \chi}{\partial \xi_\alpha}$$

and

$$\varepsilon_{ij} = \rho \frac{\partial \chi}{\partial \sigma_{ij}}$$

A sufficient condition for the existence of a generalized potential is that each of the rate functions depends on the stress only through its own conjugate thermodynamic force:

$$\dot{\xi}_\alpha = \frac{\partial \Omega_\alpha}{\partial p_\alpha}$$

and the generalized potential:

$$\Omega(\mathbf{p}, T, \xi) = \sum_\alpha \Omega_\alpha(p_\alpha, T, \xi)$$

for mathematical reasons is usually assumed to be a convex function of **p**.

The physics of plasticity

The plastic materials can change their shape by the forces and retain their new shape upon removal of such forces. The shaping process deformations are often accompanied by very slight, if any, volume changes.

Tension test
- rapid,
- requires simple apparatus,
- the preferred method of determining the material properties of metals, glass, hard plastics, textile fibers, biological tissues, and many others.

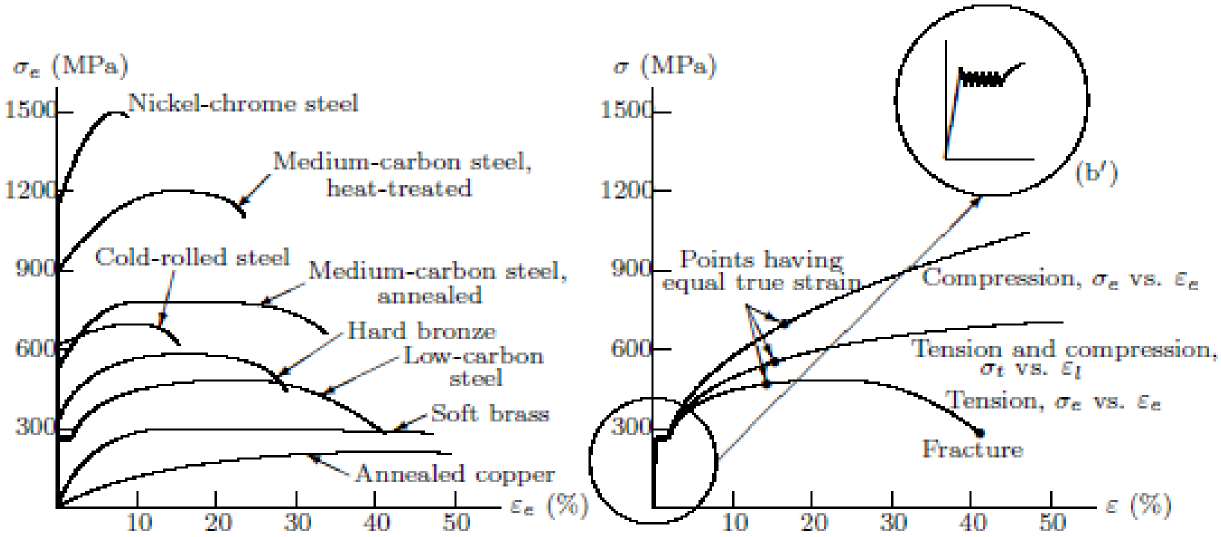

Fig. 1.4 Nominal and real stress-strain diagrams[1]

Moreover, during loading-unloading-reloading process, we observe other phenomena as aging, the Bauschinger effect, annealing or recovery.

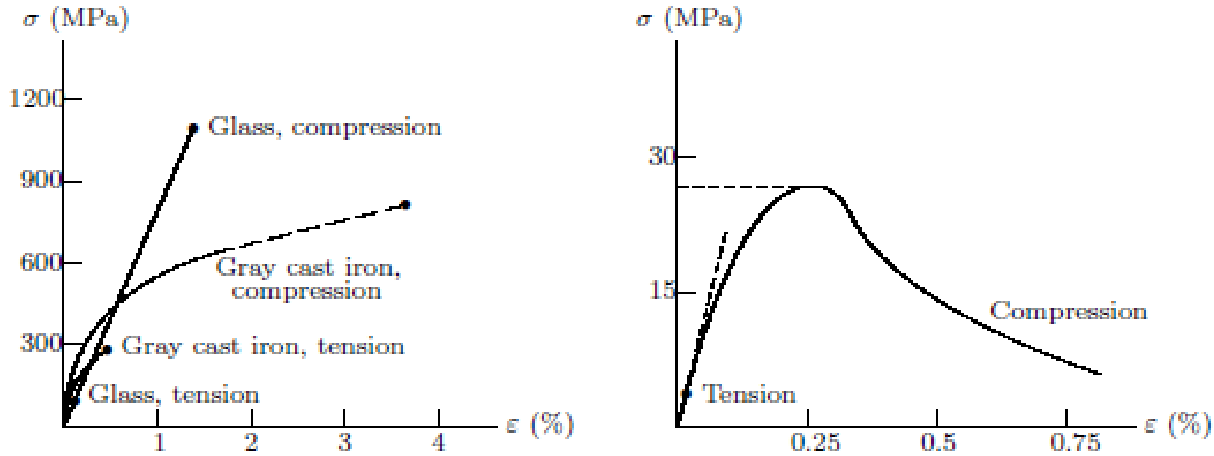

Fig. 1.5 Stress-strain diagrams for brittle solids and for concrete (and many rocks)[2]

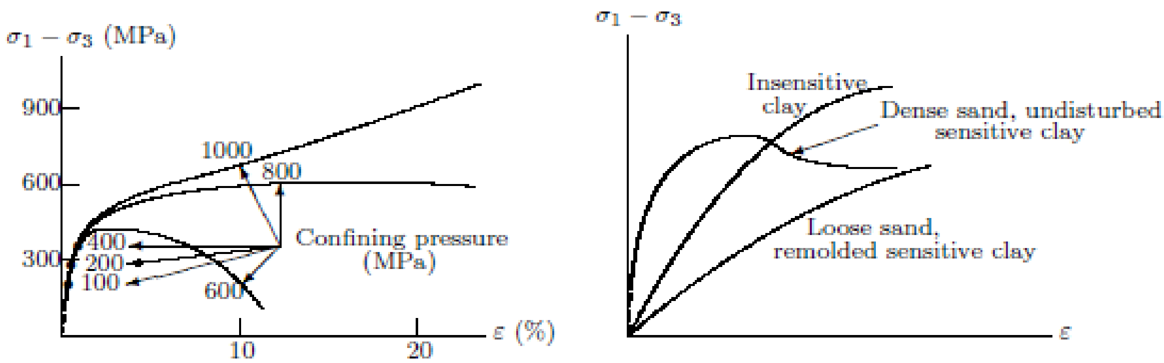

Fig. 1.6 Diagrams for rock (limestone) and soils in triaxial compression[3]

Rate effects can distort the results. They are negligible if the time taken for the test is either very long or very short compared with the characteristic time t_d of the material.

---

[1] from Lubliner
[2] form Lubliner
[3] from Lubliner

Even in the absence of the rate effects, it is not always easy to determine an accurate value for the elastic or proportional limit.
For design purposes, the conventional “yield strength” is defined as the value of the “offset” or conventional permanent strain.

Plastic deformation, schematizations

What is new?

– partial or total irreversibility of some processes (residual strains, greater stresses in elastic regime due to strain or work hardening, strain softening of compressed concrete)
– dependence of the material behavior on process history
– it is clear that the description of real mechanical properties of material should be evolutionary (material constant are not really constant, there is a virgin curve and processed curve)

Models of exponential scheme

Nonlinear exponential schemes for the materials with hardening:

– exponential plastic hardening

$$\sigma = k\varepsilon^n$$

$$\frac{\sigma}{\sigma_0} = \left(\frac{\varepsilon}{\varepsilon_0}\right)^n, \quad (0 \leq n \leq 1)$$

– rigid-plastic materials with exponential hardening (Ludwik, 1909)

$$\sigma = \sigma_0 + k\varepsilon^n \sigma = \sigma_0 + k\varepsilon^n$$

– elastic-plastic exponential hardening

$$\sigma = \begin{cases} E\varepsilon & (\varepsilon \leq \sigma_0/E) \\ k\varepsilon^n & (\varepsilon \geq \sigma_0/E) \end{cases} \sigma = \begin{cases} E\varepsilon & (\varepsilon \leq \sigma_0/E) \\ k\varepsilon^n & (\varepsilon \geq \sigma_0/E) \end{cases}$$

– elastic-plastic exponential hardening (Ramberg-Osgood, 1943)

$$\varepsilon = \frac{\sigma}{E} + k\left(\frac{\sigma}{E}\right)^n, \quad (n \geq 1)$$

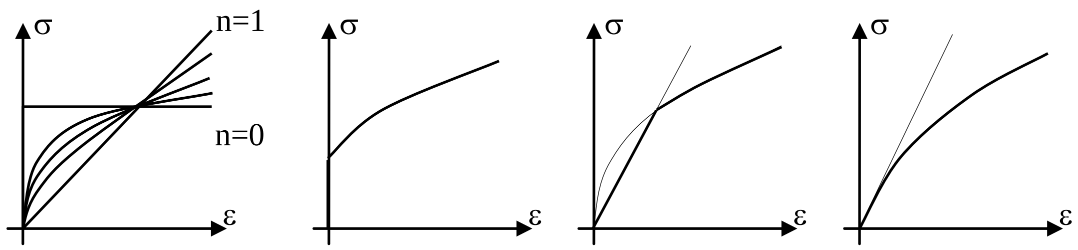

Fig. 1.7 Models: nonlinear, Ludwik, elastic-plastic and Ramberg-Osgood

Models of asymptotic ideal plasticity

There are the schemes of the type $\varepsilon = f(\sigma)$: $f(\sigma_0) = \infty$:

– two parameter scheme of hyperbolic tangent (Prager, 1938)

$$\varepsilon = \frac{\sigma_0}{E}\tanh^{-1}\left(\frac{\sigma}{\sigma_0}\right)$$

– three parameter scheme of Ylinen

$$E_t = \frac{d\sigma}{d\varepsilon} = E\frac{\sigma_0 - |\sigma|}{\sigma_0 - c|\sigma|}$$

or

$$\varepsilon = \frac{1}{E}\left[c\sigma - (1-c)\sigma_0 \ln\left(1 - \frac{\sigma}{\sigma_0}\right)\right], \quad (0 \le c \le 1)$$

– three parameter scheme of Życzkowski

$$\varepsilon = \frac{\sigma}{E\left(1 - \frac{\sigma}{\sigma_0}\right)^n}, \quad (n \ge 0)$$

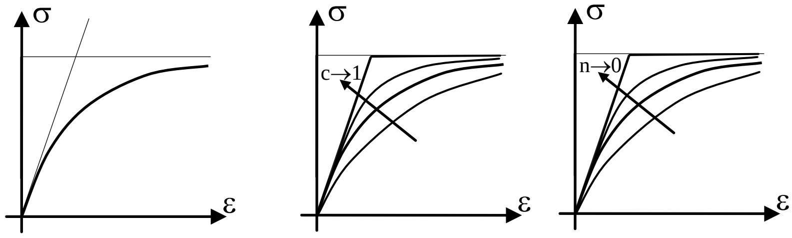

Fig. 1.8 Schemes of asymptotic plasticity (Prager, Ylinen, Życzkowski)

The Maxwell and Voigt-Kelvin relations also describe creep, it means continuing deformation at constant stress. The creep can be either bounded or steady.

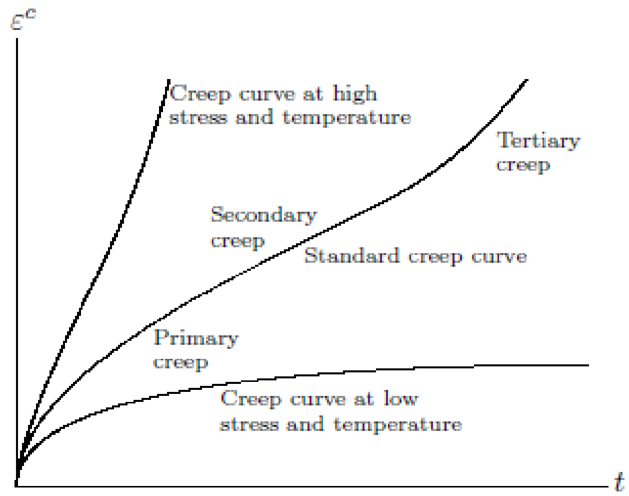

Fig. 1.9 Typical creep curves for metals [4]

Geomaterials plasticity

Soils are particulate (composed of many small solid particles from 1μm to a few mm). They have so-called void ratio (void volume), that can be (partially) saturated or dry (but still contain adsorbed water). These materials can be:

- cohesive soils
- frictional soils

Another phenomenon of the seepage of water from the voids and decrease in void ratio is known as consolidation.

The dependence between shear and normal stress in dry cohesionless soils is due to Coulomb law:

$$\tau = \sigma \tan \varphi$$

where φ is the angle of internal friction.

In wet cohesionless soils, the applied stress is the sum of the effective stress and the neutral stress due to water pressure and possibly capillary tension.

[4] from Lubliner

In clays, the shear strength is given by:

$$\tau = c + \sigma \tan \varphi$$

where c is material cohesion (shear strength under zero normal stress).
Rocks and concrete attain their ultimate strength after developing permanent strains that are significantly greater than the elastic strains (however small in absolute terms). The permanent deformation is due to several mechanisms, the foremost of which is the opening and closing of cracks. These materials exhibit strain-softening: a gradual decrease in strength with additional deformation. In the sufficiently great confining pressure, the brittle behavior of concrete and some rocks (marble, limestone) is replaced by ductility with work-hardening. The volume increase resulting from the formation and growth of cracks parallel to the direction of the greatest compressive stress is known as dilatancy[5]. The same term is applied to the swelling of dense granular soils with different causing mechanism. The uniaxial tensile strength of rock and concrete is typically between 6 and 12% the uniaxial compressive strength.

[5] don't confound with dilatation