Constitutive theory

From previous lecture you saw that "yielding" is the most striking phenomenon of plastic behavior.
However, the existence of a well-defined yield stress is rather exceptional and most materials, like hardened steel, don't exhibit distinct yielding stress.

Internal variable theory of viscoplasticity

ξ - the array of internal variables
A continuous function $f(\sigma, T, \xi)$ such that there exists a region in the space of stress components in which at given values of T and ξ, f<0 and the inelastic strain-rate tensor vanishes in that region but not outside it

$$\exists f(\sigma, T, \xi): \quad \exists \Omega_{\Sigma}: f(\sigma, T, \xi) < 0 \;\;\&\;\; \dot{\varepsilon}^{i} = 0$$

then the region constitutes the so-called elastic range, and f( ) = 0 defines the yield surface in stress space.
The orientation of the yield surface is defined in such a way that the elastic range forms its interior.
Material having such function is viscoplastic in the stricter sense.
The definition does not entail the simultaneous vanishing of all the internal variable rates $\dot{\xi}$ in the elastic region. There are different effects in the elastic region, like strain ageing, it means an evolution of the local structure while the material is stress-free. But because the phenomenon is of the order of hours, for the sake of simplicity we assume that <u>all</u> internal-variable rates vanish in the elastic region.
So, we can use a scalar function Φ such that:

$$\langle \Phi(f) \rangle = \begin{cases} 0 \text{ for } f \leq 0 \\ \Phi(f) \text{for } f > 0 \end{cases}$$

The dependence of the yield function *f* on the internal variables ξ_{α} describes what are usually called the hardening properties of the material.

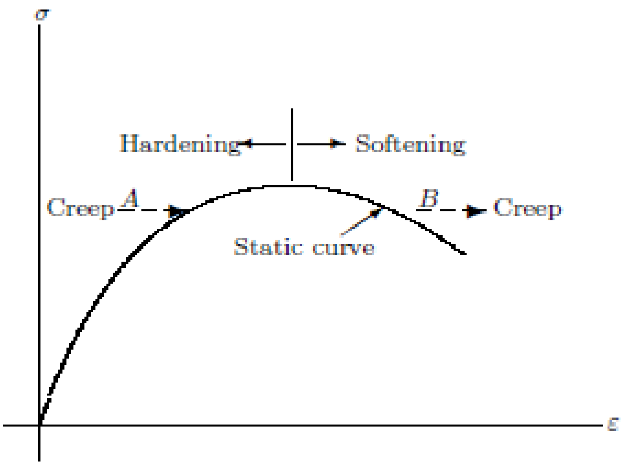

Fig. 2.1 Static curve[1]

[1] from Lubliner

Let's look on the tensile test diagram in the form of static stress-strain curve, which shows rising and falling portions, it means "hardening" and "softening" portions respectively.
If the material is viscoplastic, then its behavior is elastic at points below the curve and viscoplastic at points above the curve - it means, that the curve represents the yield surface.
If the stress is held constant above the static curve, creep occurs with increasing strain as shown by the dashed horizontal lines.
If the initial point is:

- like A, above the rising portion of the static stress-strain curve, the creep tends toward the static curve and creep is bounded
- like B, above the falling portion, the creep tends away from the static curve and creep is unbounded

We can now generalize these results from uniaxial case:

- creep toward the yield surface means hardening and yield function decreases, $\dot{f} < 0$
- creep outward the yield surface means softening with the $\dot{f} > 0$.

There are two phenomena: hardening and softening materials – the both mean the change of material behavior due to strain process

Hardening and softening in viscoplasticity

We can write down the rate of the yield function $\dot{f}$ as a sum of its derivatives with respect of internal variables:

$$\dot{f}|_{\sigma=const,T=const} = \sum_{\alpha} \frac{\partial f}{\partial \xi_\alpha} \dot{\xi}_\alpha = \phi \sum_{\alpha} \frac{\partial f}{\partial \xi_\alpha} h_\alpha = -\phi H$$

where ϕ is a scalar function containing the rate and yielding characteristics of the material. We introduce the capital H by definition equal to:

$$H = -\sum_{\alpha} \frac{\partial f}{\partial \xi_\alpha} h_\alpha$$

and from this we have:
$H > 0$ for hardening, $H < 0$ for softening materials, $H = 0$ describes a perfectly plastic material (it means the yield function is independent of internal variables ξ_α).

We introduce the definition of flow potential as a sum of partial derivatives of inelastic strain over the internal variables, times h_α:

$$h_{ij} = \sum_{\alpha} \frac{\partial \varepsilon_{ij}^i}{\partial \xi_\alpha} h_\alpha$$

then, the flow equation is expressed with use of the scalar function ϕ:

$$\dot{\varepsilon}_{ij}^i = \phi h_{ij}$$

If there exists a function g, continuously differentiable with respect to σ, wherever $f > 0$, the g is called a viscoplastic potential:

$$h_{ij} = \frac{\partial g}{\partial \sigma_{ij}}$$

Many researchers, including Perzyna, have assumed that the viscoplastic potential g is proportional to the yielding function f. This is of no great significance in viscoelasticity, but becomes highly important in rate-independent plasticity.
There were presented several propositions of viscoplastic potential. Some limiting case of great interest is a case of vanishing viscosity.
The flow equations indicate that the rate of a process with inelastic deformation increases with distance from the yield surface. If a process is very slow, then it takes place very near but just outside the yield surface.
When f remains equal to zero (or a very small positive constant):

$$\dot{f} = \frac{\partial f}{\partial \sigma_{ij}}\dot{\sigma}_{ij} + \sum_{\alpha} \frac{\partial f}{\partial \xi_{\alpha}} \phi h_{\alpha} = 0$$

with, by definition

$$\hat{f} = \frac{\partial f}{\partial \sigma_{ij}}\dot{\sigma}_{ij}$$

we have, combining the previous equations:

$$\dot{f} = \hat{f} - \phi H = 0$$

for hardening (H >0), with $\phi > 0$ above equation is possible if and only if $\hat{f} > 0$. The condition is called loading.
We get:

$$\phi = \frac{1}{H} < \hat{f} >$$

therefore:

$$\dot{\xi}_{\alpha} = \frac{1}{H} < \hat{f} > h_{\alpha}$$

In the last equation, both sides are derivatives with respect to time, so a change in time scale does not affect the equation. So, the equation is rate-independent.

When the equation is valid over a sufficiently wide range of loading rates, the behavior of such material is called rate-independent plasticity, or inviscid plasticity or simply plasticity.
Henceforth, we consider all processes to be infinitely slow, compared with the material relaxation time.

Rate-independent plasticity

Flow rule and work-hardening

Instead of inelastic strain rate we use now plastic strain and the flow equation may be written as:

$$\dot{\varepsilon}_{ij}^{p} = \dot{\lambda} h_{ij}$$

where

$$\dot{\lambda} = \begin{cases} \frac{1}{H} < \hat{f} > , & f = 0, \\ 0, & f < 0, \end{cases}$$

with:

$$H = -\sum_{\alpha} \frac{\partial f}{\partial \xi_\alpha} h_\alpha$$

The rate equations analogously become:

$$\dot{\xi}_\alpha = \dot{\lambda} h_\alpha$$

If $\frac{\partial f}{\partial \xi_\alpha} \equiv 0$, the material is called perfectly plastic. In this case $H = 0$, but $\hat{f} = \dot{f}$, and therefore the condition $\hat{f} > 0$ is impossible. Plastic deformation then occurs only if $\left(\partial f / \partial \dot{\sigma}_{ij}\right) = 0$ (neutral loading), and the definition of $\dot{\lambda}$ cannot be used. Instead it is an indeterminate positive quantity when $f = 0$ and $\left(\partial f / \partial \dot{\sigma}_{ij}\right) = 0$, and is zero otherwise.

In either case, $\dot{\lambda}$ and f can easily be seen to obey the Kuhn-Tucker conditions of optimization theory:

$$\dot{\lambda} f = 0, \dot{\lambda} \geq 0, f \leq 0.$$

Deformation theory

A deformation or total-strain theory (Hencky 1924): the plastic strain tensor itself is assumed to be determined by the stress tensor, provided that the yield criterion is met and unloading and reloading are elastic.

Proportional or radial loading – a loading in which the ratios among the stress components remain constant, provided the yield criterion and flow rule are sufficiently simple (for example, the Mises yield criterion and the flow rule with $h_{ij} = s_{ij}$). It's been proved that the stress states derived from the deformation and the incremental flow theories converge if:

- the deformation develops in a definite direction, or
- a material whose yield surface has a singular point or corner and the stress point remains at the corner in the course of loading.

Work-Hardening

It has been already said that: H>0 means hardening, H=0 means perfect plasticity and H<0 means softening.

When we draw the yield function in the stress-space, we see that H>0 implies that, at least locally, the yield surface is expanding (in the stress space).

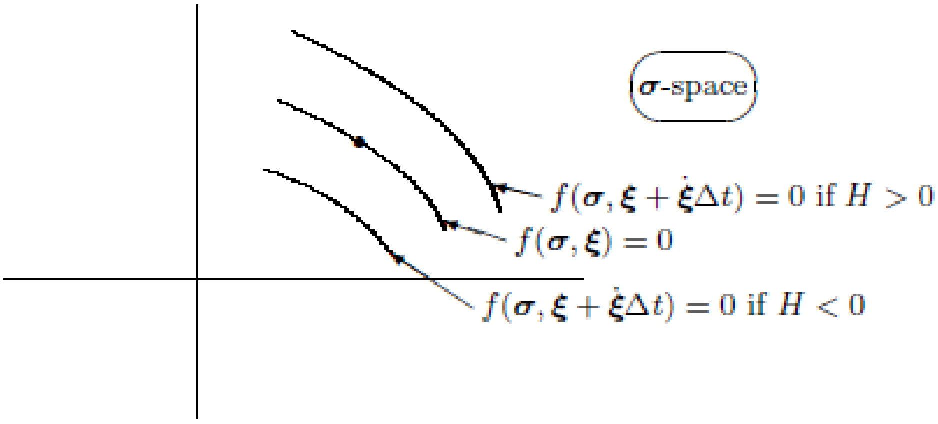

Fig. 2.2 Hardening and softening in rate-independent plasticity[2]

A contracting yield surface denotes work-softening. A stationary yield surface means perfect plasticity.
The yield function may be a function of stress state, plastic strains and a hardening parameter:

$$f(\sigma, \varepsilon^p, \kappa) = F\big(\sigma - \rho(\varepsilon^p)\big) - k(\kappa),$$

describes both isotropic and kinematic hardening. The hardening is:

- isotropic if $\rho \equiv 0$ and $dk/d\kappa > 0$, and
- purely kinematic if $dk/d\kappa \equiv 0$ and $\rho \neq 0$,
- perfectly plastic if $dk/d\kappa \equiv 0$ and $\rho \equiv 0$.

Drucker's Postulate

A more restricted definition of work-hardening was formulated by Drucker in 1950 and 1951 by generalizing the characteristics of uniaxial stress-strain curves. With a single stress component σ and the conjugate plastic strain rate $\dot{\varepsilon}^p$:

$$\dot{\sigma}\dot{\varepsilon}^p \begin{cases} \geq 0, & \text{hardening material} \\ = 0, & \text{perfectly plastic material} \\ \leq 0, & \text{softening material} \end{cases}$$

When the above inequalities hold equally well when they are multiplied by the infinitesimal time increment *dt*, so for $d\sigma d\varepsilon^p$. This product has the dimensions of work per unit volume.
Drucker defines a "stable" plastic material as one in which the work done during incremental loading is positive, and during loading-unloading is nonnegative.
The definition can be extended to general three-dimensional states of stress and strain. Drucker's inequality is valid for both work-hardening and perfectly plastic materials:

$$\dot{\sigma}_{ij}\dot{\varepsilon}_{ij}^p \geq 0$$

Very common interpretation of Drucker's postulate is that in terms of work. Often, it is referred to as a quasi-thermodynamic postulate, but it should be clearly stated, that it is independent of the basic laws of thermodynamics.
Because the left part of inequality represents the product, it can be said that the plastic strain rate cannot oppose the stress rate.

[2] from Lubliner

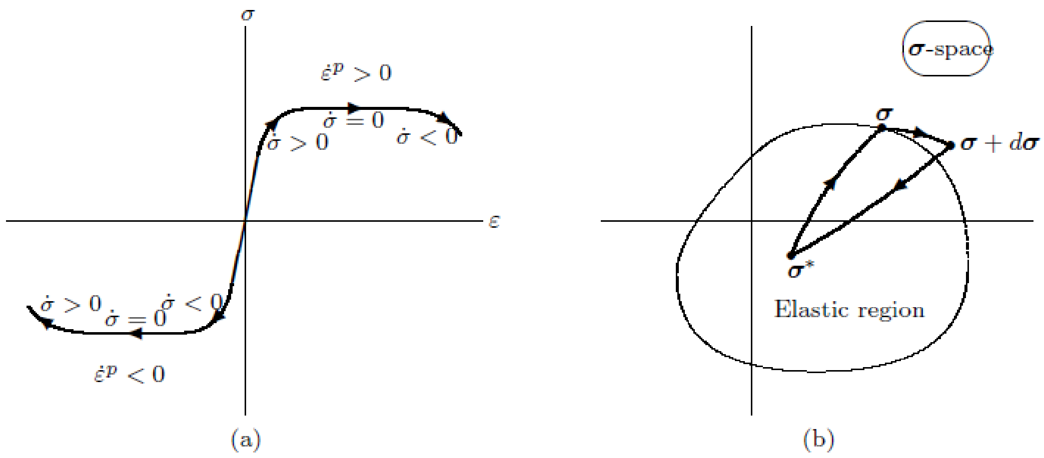

Fig. 2.3 Drucker's postulate: (a) in uniaxial stress-strain plane;(b) in stress space[3]

Drucker's postulate holds for any stress increment, not necessary a small one:

$$(\sigma_{ij} - \sigma_{ij}^{*})\dot{\varepsilon}_{ij}^{p} \geq 0$$

Maximum-Plastic-Dissipation Postulate and Normality

The above equation is a necessary condition for Drucker's postulate, but it is not a sufficient one. Using its uniaxial counterpart:

$$(\sigma - \sigma^{*})\dot{\varepsilon}^{p} \geq 0$$

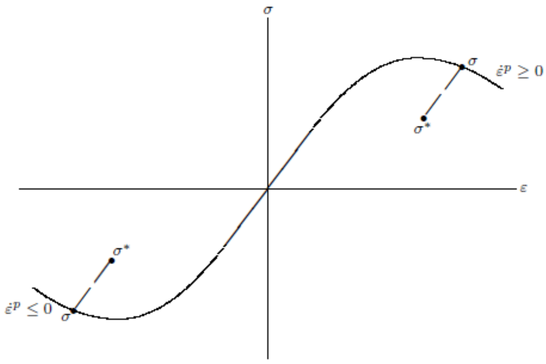

Fig. 2.4 Maximum-plastic-dissipation postulate (in uniaxial stress-strain plane)

It can be seen that work-softening and perfectly plastic materials have this property as well. It is called the postulate of maximum plastic dissipation.

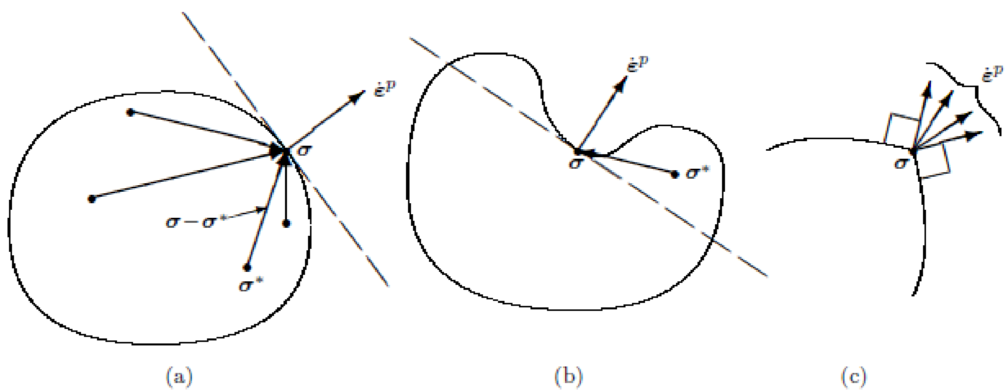

Fig. 2.5 Yield surface with associated flow rule: (a) normality; (b) convexity; (c) corner

[3] from Lubliner

The postulate has consequences of the highest importance in plasticity theory. Consequences:

- normality rule: when we "attack" the yielding surface from different points in the elastic region, we see that the plastic strain rate vector should be directed along the outward normal there
- convexity rule: when the rate of plastic strain is normal to the yield surface, to keep the inequality valid, the yield surface should be convex; in other words, the entire elastic region must lie to one side of the tangent
- the outward normal vector is proportional to the gradient of the yield surface $f(\sigma,\xi)=0$. (excluding the points of singularity that can be treated separately)

Let us define the plastic dissipation (the maximum being taken over all σ^*:

$$D_p(\dot{\varepsilon}^p;\xi)=\max_{\sigma*}\sigma_{ij}^*\dot{\varepsilon}_{ij}^p$$

We get the principle of maximum plastic dissipation:

$$D_p(\dot{\varepsilon}^p;\xi)\geq\sigma_{ij}^*\dot{\varepsilon}_{ij}^p$$

Because the gradient of smooth yield surface *f* (in stress space) is proportional to the outward normal vector, the normality rule may be expressed as:

$$h_{ij}=\frac{\partial f}{\partial\sigma_{ij}}$$

where h_{ij} is the tensor function appearing in the flow equation,
means that the function *f* defining the yield surface is itself a plastic potential, so this is the case of a flow rule associated with the yield criterion, or, briefly, an associated (associative) flow rule. The materials obeying an associated flow rule are called standard materials in the French literature.
A nonassociated flow rule is a rule derivable from a plastic potential g that is distinct from *f*; more precisely:

$$\frac{\partial g}{\partial\sigma_{ij}}\text{ is not proportional to }\frac{\partial f}{\partial\sigma_{ij}}$$

Iliushin's Postulate

It can be shown that the normality rule follows from a "postulate of plasticity" in strain space proposed by Iliushin (1961), namely, that in any cycle that is closed in strain space,

$$\oint\sigma_{ij}d\varepsilon_{ij}\geq 0$$

where the equality holds only if the process is elastic. However Iliushin's postulate is satisfied for processes in which the original yield surface is inside all subsequent yield surfaces. The last condition is satisfied in materials with isotropic hardening, but not in general. Consequently Iliushin's postulate is a stronger (less general) hypothesis than the principle of maximum plastic dissipation.

Yield criteria, Flow Rules and Hardening Rules

There are several possibilities to graphically present the yield criteria.

Haigh-Westergaard space, Meldahl space, meridian plane

The *Haigh-Westergaard space* is the space with the set of principal stress coordinates. The axis, which is equally inclined to all axes, is *hydrostatic axis* (*mean stress axis*). The *Meldahl surface* or *deviatory surface*, or π-plane, is the surface perpendicular to the mean stress axis.
The surface passing through the hydrostatic axis is the *meridian plane*, where its angle describes angle between the meridian plane and the first principal axis. A point with $\theta = 0$ corresponds to a tensile meridian of the surface. A point with $\theta = \pi/3$ corresponds to a compressive meridian.
Another cross-section is the section by the plane $\sigma_i = 0$, for the plane state of stress.
Sometimes, the same set of coordinates $\sigma - |\tau|$, like for Mohr's circles, is the best choice to present the yielding/failure criterion.

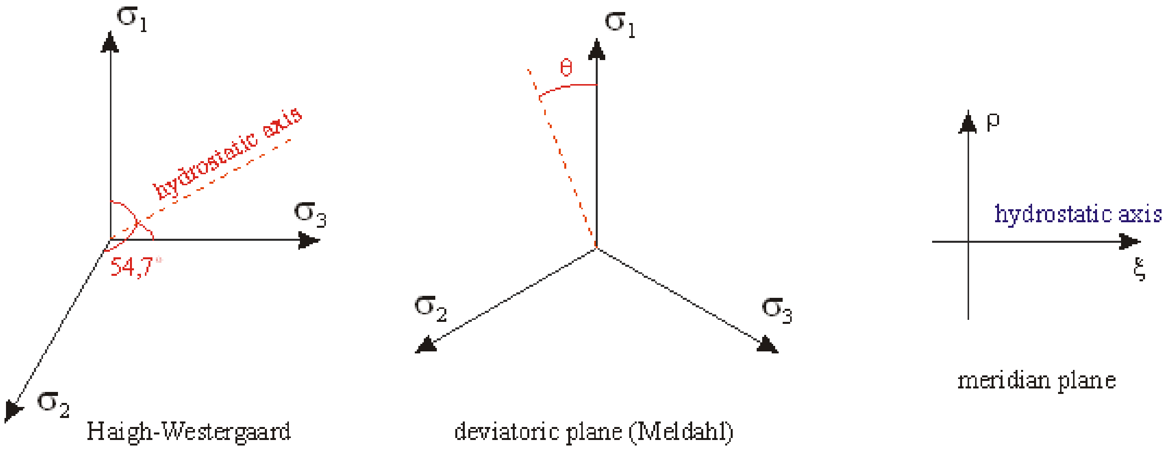

Fig. 2.6 Haigh-Westergaard space and intersection with particular plane

For instance, the Rankine criterion of failure of materials having different tensile and compression strengths (like concrete), in the Haigh-Westergaard space is a cube located eccentrically.

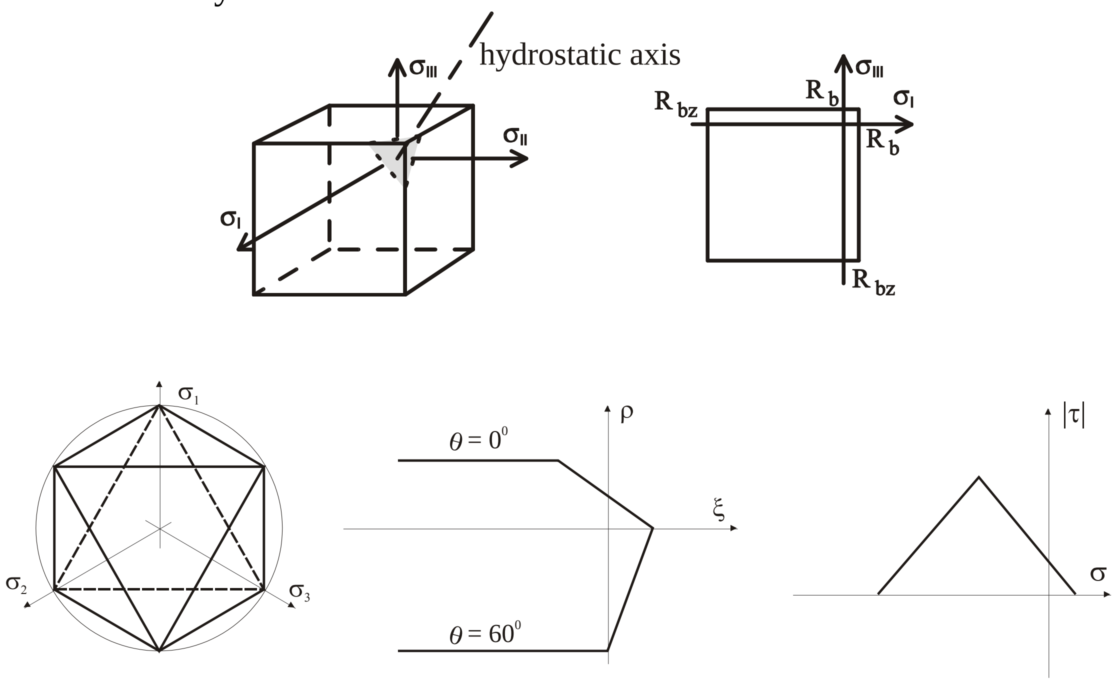

Fig. 2.7 Different views of Rankine's criterion

We can express the principal stress values in terms of the hydrostatic stress value (the distance from the origin to the deviatory plane, ξ), the length of the stress vector in the deviatory plane, ρ, and an angle of the meridian plane, θ:

$$\sigma_1 = \sigma_m + \frac{2}{\sqrt{3}}\sqrt{J_2}\cos\theta = \frac{1}{\sqrt{3}}\xi + \sqrt{\frac{2}{3}}\rho\cos\theta$$

$$\sigma_2 = \sigma_m + \frac{2}{\sqrt{3}}\sqrt{J_2}\cos\left(\theta - \frac{2}{3}\pi\right) = \frac{1}{\sqrt{3}}\xi + \sqrt{\frac{2}{3}}\rho\cos\left(\theta - \frac{2}{3}\pi\right)$$

$$\sigma_3 = \sigma_m + \frac{2}{\sqrt{3}}\sqrt{J_2}\cos\left(\theta + \frac{2}{3}\pi\right) = \frac{1}{\sqrt{3}}\xi + \sqrt{\frac{2}{3}}\rho\cos\left(\theta + \frac{2}{3}\pi\right)$$

where $0 \le \theta \le \frac{\pi}{3}$

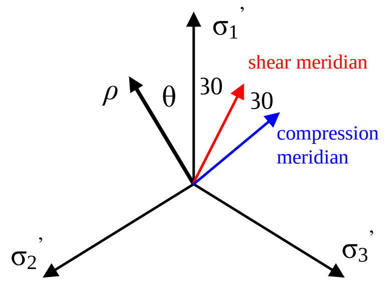

Fig. 2.8 Shear and compression meridians

Yield criteria independent of the means stress

Tresca criterion

From the assumption that plastic deformation occurs when the maximum shear stress over all planes attains a critical value (of the current yield stress in shear), the criterion may be represented by the yield function:

$$f(\sigma,\xi) = \frac{1}{2}\max(|\sigma_1 - \sigma_2|, |\sigma_2 - \sigma_3|, |\sigma_3 - \sigma_1|) - k(\xi)$$

or, equivalently

$$f(\sigma,\xi) = \frac{1}{4}(|\sigma_1 - \sigma_2|, |\sigma_2 - \sigma_3|, |\sigma_3 - \sigma_1|) - k(\xi)$$

The projection of the Tresca yield surface in the π-plane is a regular hexagon, while in the $(\sigma_1 - \sigma_3)(\sigma_2 - \sigma_3)$-plane it takes the form of the irregular hexagon shown in Fig. 2.9

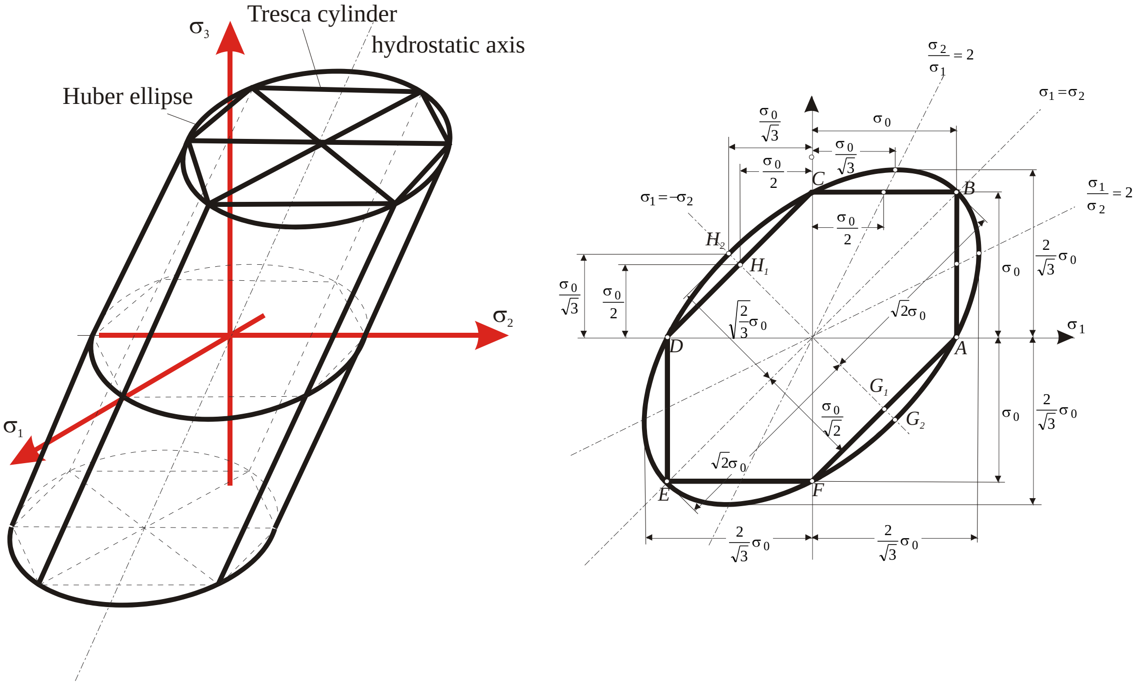

Fig. 2.9 Comparison of Tresca and Huber-Mises criteria

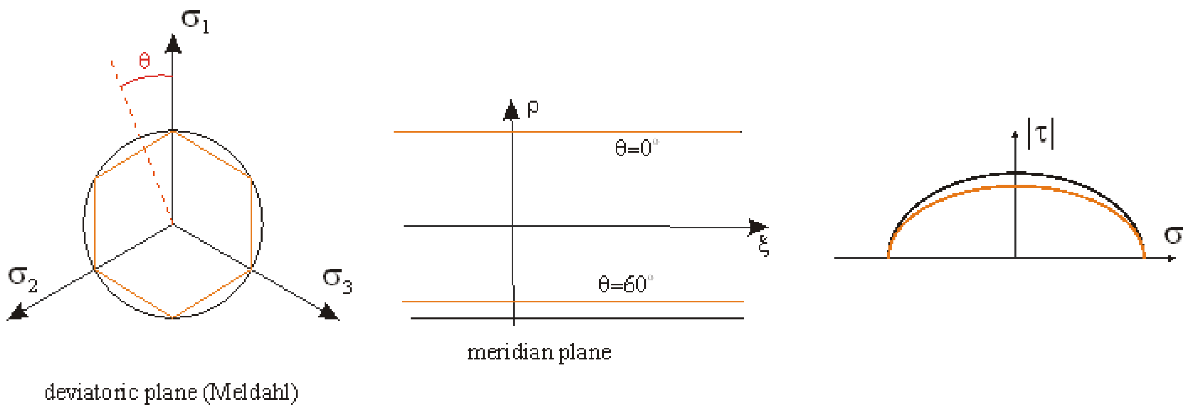

Fig. 2.10 Comparison of Tresca and Huber-Mises criteria

Tresca criterion: associated flow rule

The Tresca yield surface is singular. Its associated flow rule can be derived by means of a formal application of distribution functions:

$$\frac{d}{dx}|x| = \operatorname{sgn} x$$

where

$$\operatorname{sgn} x = 2H(x) - 1 = \begin{cases} +1, x > 0 \\ -1, x < 0 \end{cases}$$

In this way we obtain:

$$\dot{\varepsilon}_1^p = \frac{1}{4}\dot{\lambda}[\operatorname{sgn}(\sigma_1 - \sigma_2) + \operatorname{sgn}(\sigma_1 - \sigma_3)]$$

where, for work-hardening materials $\dot{\lambda} = \langle \hat{f} \rangle / H$, with

$$H = \sum_{\alpha} \frac{\partial h}{\partial \xi_\alpha} h_\alpha$$

while for the perfectly plastic material $\dot{\lambda}$ is indeterminate.

Lévy flow rule and Huber-Mises yield criterion

The general form proposed by Lévy:

$$\dot{\varepsilon}_{ij}^p = \dot{\lambda} s_{ij}$$

The yield criterion with which this flow rule is associated is the Huber-Mises criterion, represented by the yield function:

$$f(\sigma, \xi) = \sqrt{J_2} - k(\xi)$$

or, in an alternative form with the dependence on ξ not shown explicitly:

$$f(\sigma, \xi) = J_2 - k^2$$

where $k(\xi)$ is again the yield stress in shear at the current values of the variable ξ
Expressing J_2 in term of the principal stresses, the Huber-Mises criterion is:

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2 = 6k^2$$

or

$$\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_2\sigma_3 - \sigma_3\sigma_1 - \sigma_1\sigma_2 = 3k^2$$

In the π-plane the Huber-Mises yield surface is of a circle of radius $\sqrt{2}k$ and in the $(\sigma_1 - \sigma_3)(\sigma_2 - \sigma_3)$ plane of an ellipse.

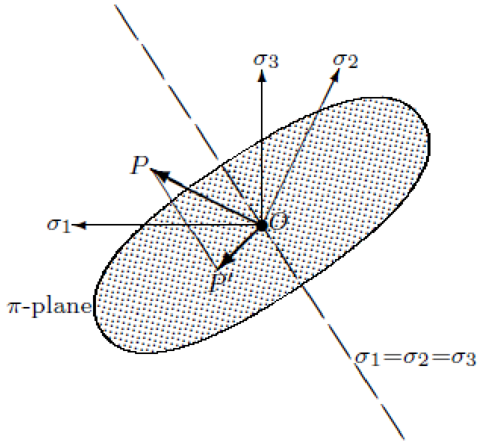

Fig. 2.11 Deviatory plane (π − plane)

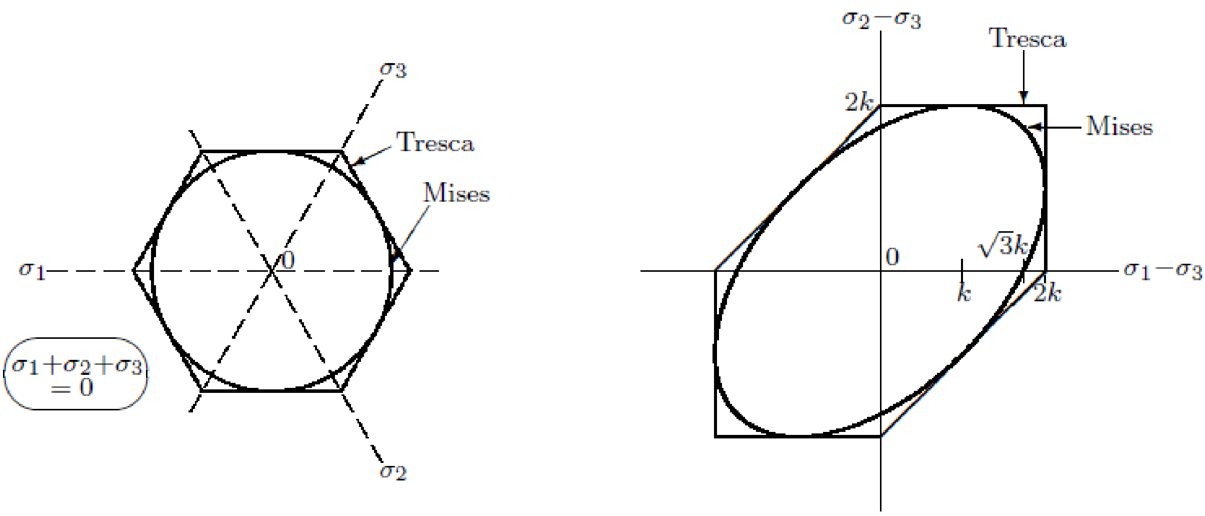

Fig. 2.12 Yield surfaces' projections

The plastic dissipation for the Huber-Mises criterion and the associated flow rule is given by:

$$D_p(\dot{\varepsilon}^p;\xi) = \sigma_{ij}\dot{\varepsilon}_{ij}^p = \dot{\lambda}s_{ij}s_{ij} = \sqrt{2J_2}\sqrt{\dot{\varepsilon}_{ij}^p\dot{\varepsilon}_{ij}^p} = k(\xi)\sqrt{2\dot{\varepsilon}_{ij}^p\dot{\varepsilon}_{ij}^p}$$

Prandtl and Reuss generalization; expressed in terms of total strain rate:

$$\dot{\varepsilon}_{kk} = \frac{1}{3K}\dot{\sigma}_{kk}$$

$$\dot{e}_{ij} = \frac{1}{2G}\dot{s}_{ij} + \dot{\lambda}s_{ij}$$

Some generalization of the yield function with the dependence on J_3 included (a typical form):

$$f(\sigma) = \left(1 - c\frac{J_3^2}{J_2^3}\right)^{\alpha} J_2 - k^2$$

where the exponent α is taken as $\frac{1}{3}$ and 1, k is as usual the yield stress in simple shear, ad c is a parameter.

Anisotropic yield criteria

Anisotropy in yielding may be of two types:

- initial anisotropy, usually in materials that are structurally anisotropic
- induced anisotropy, as a result of work-hardening process.

Hill:

$$\frac{1}{2}A_{ijkl}\sigma_{ij}\sigma_{kl} = k^2$$

where A is a fourth-rank tensor which has the same symmetries as the elasticity tensors. If the yield criterion is independent of mean stress,, then A also obeys $A_{ijkk} = 0$, so that is has at most 15 independent components; the isotropic (Huber-Mises) case corresponds to $A_{ijkl} = \delta_{ik}\delta_{jl} - \frac{1}{3}\delta_{ij}\delta_{kl}$.

Yield criteria dependent on the mean stress

The critical combination may be expressed in a form $\tau = \pm g(\sigma)$, unchanged when the direction of the shear stress is reversed. The envelopes of the Mohr's circles representing failure and are therefore called the *Mohr failure (rupture) envelopes*.

Mohr-Coulomb criterion

When the envelopes are straight lines, the criterion can be reduced to:

$$g(\sigma) = c - \mu\sigma$$

where σ is positive in compression, c is the cohesion and $\mu = \tan\phi$ is the coefficient of internal friction in the sense of Coulomb model of friction. (When $\phi = 0$ the criterion reduces to that of Tresca).

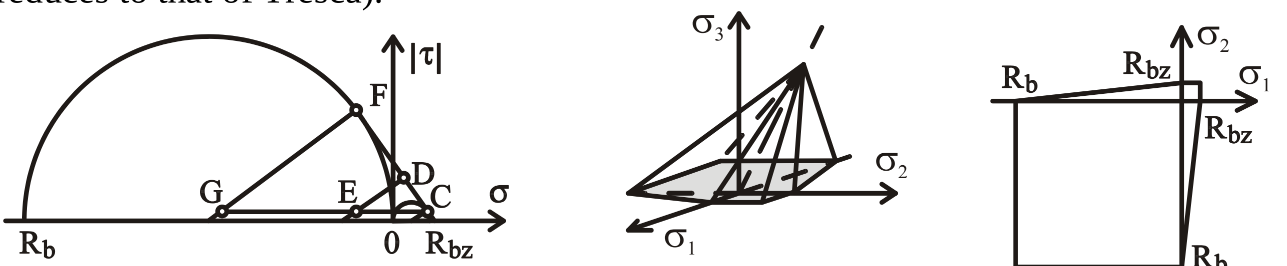

Fig. 2.13 Mohr-Coulomb criterion

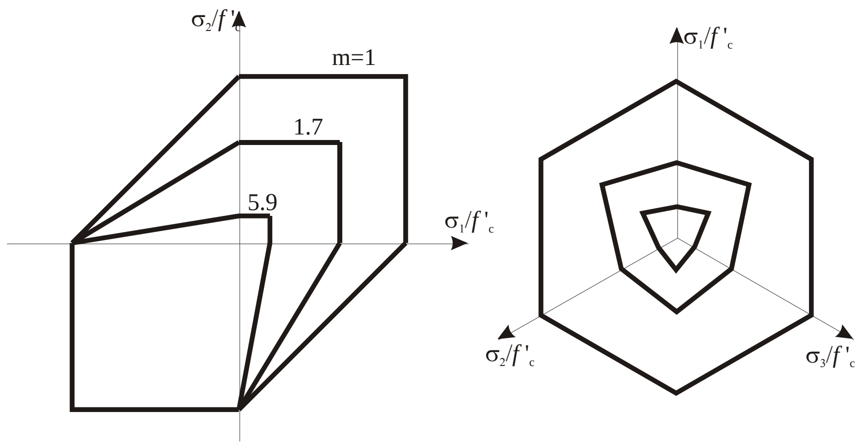

Fig. 2.14 Mohr-Coulomb criterion for different tensile/compressive yielding ratio

The associated plastic dissipation was shown by Drucker to be:

$$D_p\,(\dot{\varepsilon}^p;\xi) = c\cot\phi\,(\dot{\varepsilon}_1^p + \dot{\varepsilon}_2^p + \dot{\varepsilon}_3^p)$$

The failure surfaces in principal stress space are obviously planes that intersect to form a hexagonal pyramid:

$$\sigma_{max} - \sigma_{min} + (\sigma_{max} + \sigma_{min})\sin\phi = 2c\cos\phi$$

where σ_{max} and σ_{min} denote respectively the (algebraically) largest and smallest principal stresses. The last equation may be rewritten as

$$\sigma_{max} - \sigma_{min} + \frac{1}{3}[(\sigma_{max} - \sigma_{int}) - (\sigma_{int} - \sigma_{min})]\sin\phi = 2c\cos\phi - \frac{2}{3}I_1\sin\phi$$

where σ_{int} denotes the intermediate principal stress. The family of criteria based on Coulomb friction may be described by equation of the form:

$$\bar{F}(J_2,J_3) = c - \lambda I_1$$

where c and λ are constants.

Drucker-Prager criterion

Combining Coulomb friction with the Huber-Mises yield criterion, Drucker and Prager proposed yielding criterion which occurs on the octahedral planes when $\tau_{oct} = \sqrt{\frac{2}{3}}k - \frac{1}{3}\mu I_1$, so:

$$f(I_1,J_2) = \alpha I_1 + \sqrt{J_2} - k = 0$$

or

$$f(\xi,\rho) = \sqrt{6}\alpha\xi + \rho - \sqrt{2}k = 0\,.$$

The associated plastic dissipation is

$$D_p(\dot{\varepsilon}^p;\xi) = \frac{k\sqrt{2\dot{\varepsilon}_{ij}^p\dot{\varepsilon}_{ij}^p}}{\sqrt{1+\mu^2}}$$

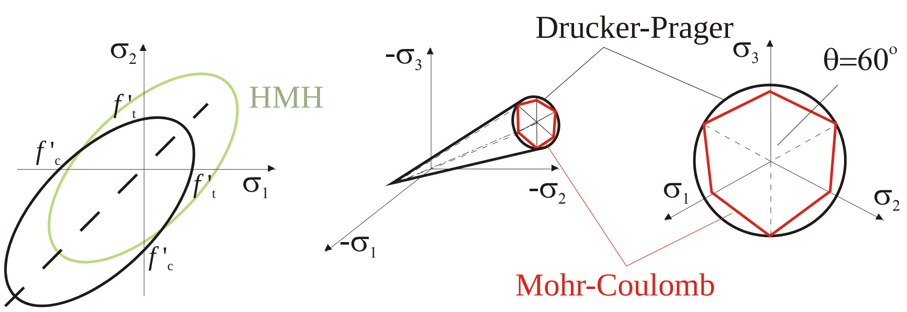

Fig. 2.15 Mohr-Coulomb and Drucker-Prager criteria

The Drucker-Prager criterion for a biaxial stress:

$$\alpha(\sigma_1+\sigma_2)+\sqrt{\tfrac{1}{3}\left(\sigma_1^2-\sigma_1\sigma_2+\sigma_2^2\right)} = k\,.$$

The Drucker criterion

The Drucker proposition is a generalization of CTG criterion:

$$(\sigma_I - \sigma_{III}) + \tfrac{1}{3}\beta I_1 - k = 0\,,$$

which in principal stress space is pyramid.

Anisotropic failure/yield criteria

The proposition of Mises with use of 4th order plasticity tensor:

$$\Pi_{ijkl}\sigma_{ij}\sigma_{kl} - k = 0$$

and 21 independent material constants is simplified by Hill into six parameters criterion in the case of orthotropy.

Hardening rules

A hardening rule it is a specification of the dependence of the yield criterion on the internal variables, along with the rate equations for these variables.

Isotropic hardening

The yield functions we have studied so far are all reducible to the form:

$$f(\sigma,\xi) = F(\sigma) - k(\xi)$$

When an internal variable is identified with the hardening variable κ, defined as either the plastic work or as the effective plastic strain, so that the work-hardening modulus *H* is

$$H = \begin{cases} k'(W_p)\sigma_{ij}h_{ij} \\ k'(\bar{\varepsilon}^p)\sqrt{\frac{2}{3}h_{ij}h_{ij}} \end{cases}$$

Work-hardening in rate-independent plasticity corresponds to a local expansion of the yield surface. The present behavior of swelling of yielding surface, called isotropic hardening, represents a global expansion, with no change in shape.

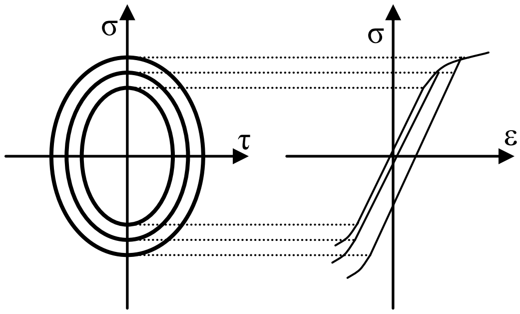

Fig. 2.16 Plastic isotropic hardening

There is no the Bauschinger's effect. The hardening depends on non-decreasing function of plastic strain

Taylor-Quinney (1931):

$$\sigma_i = f_2\left(W^p\right)$$

where the plastic strain work (dissipation energy):

$$W^p = W^d = \int_0^{\tilde{\varepsilon}^p}\sigma_{ij}d\varepsilon_{ij}^p = \int_0^{\tilde{e}^p}\sigma_{ij}de_{ij}^p = \int_0^{\tilde{e}^p}s_{ij}de_{ij}^p$$

is non-decreasing variable also.

Kinematic hardening

The yield function:

$$f(\sigma,\xi) = F(\sigma - \rho) - k(\xi)$$

If $\rho \equiv 0$ and if k depends only on κ – isotropy. This is a fairly good agreement with the Bauschinger effect for those materials whose stress-strain curve in the work-hardening range can be approximated by a straight line ("linear hardening"), and for such materials that Melan proposed the model in which $\rho = c\varepsilon^p$, with c a constant. A similar idea was also proposed by Ishlinskii (1954), and a generalization of it is due to Prager (1955, 1956), who coined the term "kinematic hardening" on the basis of his use of a mechanical model in explaining the hardening rule, see Fig.

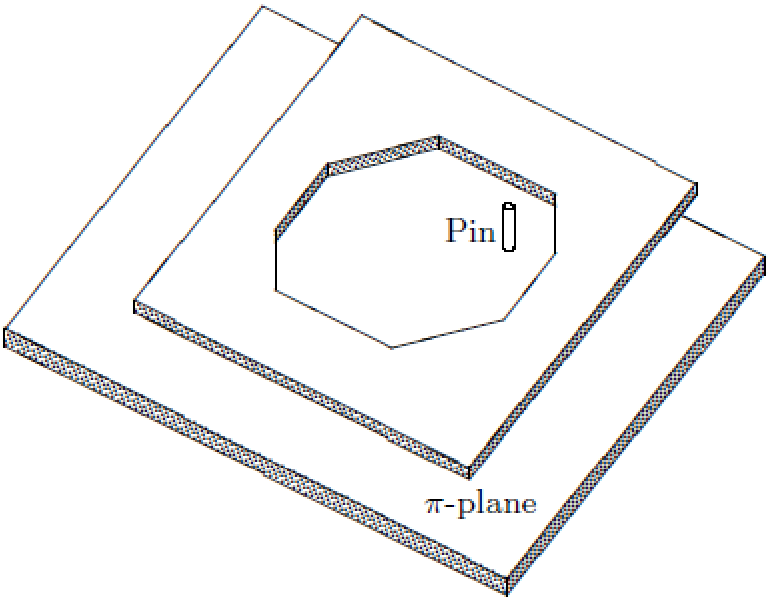

Fig. 2.17 Prager's mechanical model of kinematic hardening

A kinematic hardening model is also capable of representing induced anisotropy, since a function $F(\sigma - \rho)$ that depends only on the invariants of its argument stops being an isotropic function of the stress tensor as soon as ρ (so-called the back stress) differs from zero.
The equation

$$\rho_{ij} = c\varepsilon_{ij}^p$$

does not imply proportionality between the vectors representing ρ and ε^p in any space other than the nine-dimensional space of second-rank tensors
In more sophisticated Melan-Prager model the back stress is treated as a tensorial internal variable with its own rate equation:

$$\dot{\rho}_{ij} = c\dot{\varepsilon}_{ij}^p$$

where c not need be a constant but may depend on other internal variables.
Another example of a kinematic hardening model is that due to Ziegler (1959):

$$\dot{\rho}_{ij} = \dot{\mu}(\sigma_{ij} - \rho_{ij})$$

where

$$\dot{\mu} = \frac{\langle \frac{\partial f}{\partial \sigma_{kl}} \dot{\sigma}_{kl} \rangle}{\frac{\partial f}{\partial \sigma_{mn}} (\sigma_{mn} - \rho_{mn})}$$

in order to satisfy the consistency condition $\dot{f} = 0$.

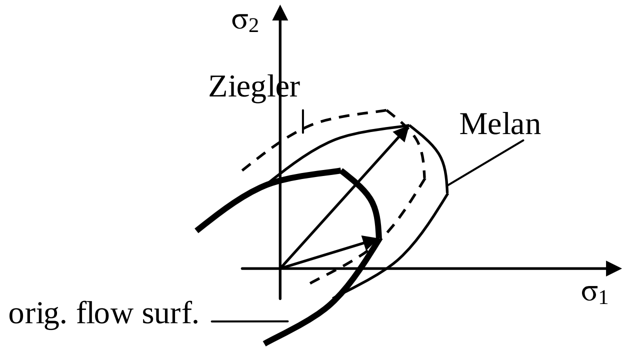

Fig. 2.18 Translation of the plastic flow surface

Generalized hardening rules

Other models:

- combined hardening by Hodge
- coined hardening by Chaboche
- with family of the back stresses by Mróz (stress-strain curves are piecewise linear)
- two-surface model by Dafalias (with bounding/loading/memory surface – an original and "final" flow surface)
- anisotropy distortion in initially isotropic materials (Sawczuk, with decomposition of fourth rank back stress tensor into elastic and plastic parts)

The mixed hardening is a combination of isotropic and kinematic hardening.
The anisotropic hardening consists on the active surface change. There are a few kinds of such hardening:

- general type
- with independent mechanisms
- qualitative change of yielding surface (plastic corners)

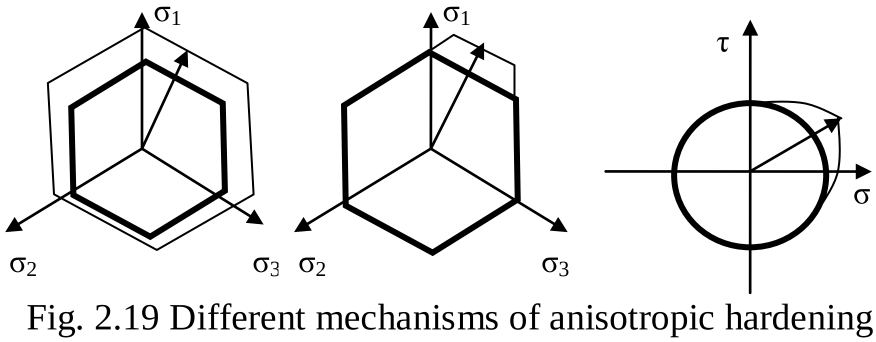

Fig. 2.19 Different mechanisms of anisotropic hardening