Plasticity of bar structures

Limit capacity of the cross-section

We assume the cross-section has one (vertical) symmetry axis. Moreover:

– uniaxial state of stress $\sigma = \begin{cases} E\varepsilon, & \text{dla} \quad \sigma < R_e \\ \pm R_e \end{cases}$

– Bernoulli's hypothesis of plane cross-sections.

We use the conditions of equilibrium:

$$\iint \sigma dA = N,$$

$$\iint \sigma z dA = M$$

The stages of the cross-section work are:

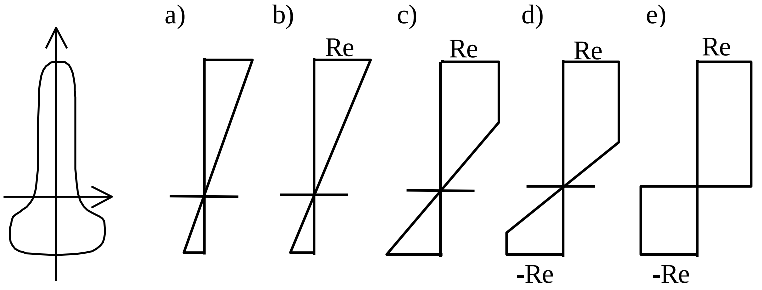


Fig. 3.1 Stages of the cross-section work

– *elastic range*, the principle of superposition is valid due to linearity of equations

– yielding of the first extreme fibers, it is *elastic limit* of the cross-section bearing capacity,

– one-sided yielding in elastic-plastic range,

– two-sided yielding, the elastic-plastic range,

– full yielding with the plastic hinge arising, it is *a plastic limit* of the cross-section bearing capacity.

The elastic-plastic range calculations are the most complicated. The problem consists on determination of two process parameters from the equations of equilibrium. There are two regions in the cross-section: elastic and plastic, divided by the *plasticity front*. The position of the front, the integral limits, is unknown.

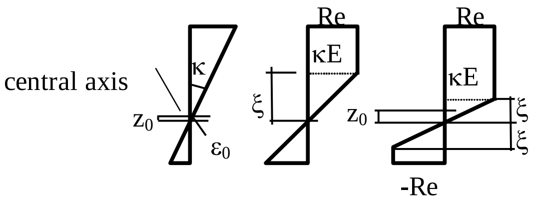


Fig. 3.2 Parameters of the stress distribution

From Bernoulli's hypothesis, we have:

$$\varepsilon = \varepsilon_0 + \kappa z\,.$$

Two parameters determine the strain distribution: the bar axis strain and its curvature. Sometimes, it is the easier way to apply other parameters, like:

– position of neutral axis

$$\varepsilon(z_0)=0 \quad \Rightarrow \quad z_0=-\frac{\varepsilon_0}{\kappa},$$

- position of the plastic front

$$\varepsilon_0+\kappa z_p=\pm R_e/E \quad \Rightarrow \quad z_p=\pm\frac{R_e}{E\kappa}-\frac{\varepsilon_0}{\kappa},$$

- range of the elastic zone

$$\xi\kappa E=\pm R_e \quad \Rightarrow \quad \xi=\frac{R_e}{E\kappa}$$

Only two parameters are independent.

Elastic and plastic limit capacity of the cross-section

> *Elastic limit capacity of the cross-section is the set of cross-section forces causing yielding of the (first) extreme fibers.*

For simple bending, the value of the bending moment causing the first yielding can be easily calculated from the equation of elastic range (Hooke's equations).

> *Plastic limit capacity of the cross-section is the set of cross-section forces causing full yielding of the whole cross-section.*

For simple bending the plastic limit value can be calculated from rectangular distribution of the normal stress. From the first condition of zero normal force we have:

$$\iint \sigma dF=0, \quad \Rightarrow \quad \iint_{A_1} R_e dF+\iint_{A_2}(-R_e)dF=0, \quad \Rightarrow \quad A_1=A_2$$

so, the neutral axis halved cross-section (divides cross-section into two equal areas). From the second condition we get that the plastic limit value of the bending moment is equal to the sum of static moments of the cross-section halves.

$$\overline{\overline{M}}=\iint \sigma z dF \quad \Rightarrow \quad \iint_{A_1} R_e z dF-\iint_{A_2} R_e z dF=R_e(S_{A1}-S_{A2})=R_e\left(|S_{A1}|+|S_{A2}|\right).$$

The condition of equilibrium is valid in every coordinate set. If one axis is the principal central axis, the static moments differ only by their signs, so the formula may be rewritten:

$$\overline{\overline{M}}=2R_e\left|S^0{}_{A1,A2}\right|.$$

Similarly to elastic cross-section factor, we introduce the plastic cross-section factor as:

$$\overline{\overline{W}}\equiv 2\left|S^0{}_{A1,A2}\right|.$$

Examples

triangular cross-section b×h

$$W_{\mathrm{el}}=\frac{bh^3}{36}\frac{3}{2h}=\frac{bh^2}{24}$$

the neutral axis: $A_1=\tfrac{1}{2}A=\tfrac{1}{4}bh \quad \Rightarrow \quad h_1=\dfrac{h}{\sqrt{2}}, \quad b_1=\dfrac{b}{\sqrt{2}}$, so:

$$\overline{\overline{W}}=2S_{A1}=2\tfrac{1}{2}b_1h_1\left(\tfrac{2}{3}h-\tfrac{2}{3}h_1\right)=\frac{2-\sqrt{2}}{6}bh^2$$

elastic limit $\overline{M}=M_{\max} \quad \Rightarrow \quad R_e\dfrac{bh^2}{24}=\tfrac{8}{81}ql^2 \quad \Rightarrow \quad \bar{q}=\tfrac{27}{64}\dfrac{bh^2}{l^2}R_e$

plastic limit $\overline{\overline{M}} = M_{\max} \quad \Rightarrow \quad R_e \dfrac{2-\sqrt{2}}{6} bh^2 = \tfrac{8}{81} ql^2 \quad \Rightarrow \quad \overline{\overline{q}} = \tfrac{27}{16}\left(2-\sqrt{2}\right)\dfrac{bh^2}{l^2} R_e$

Generalized stress

A concept of cross-section forces (axial force and bending moment) as generalized stress and axis elongation with its curvature as generalized strains is of great usefulness in the limit analysis of structures (see below).
Generalized stress may coincide with the actual stresses, or they may be local stress resultants integrated over one or two dimensions, or even a whole finite element of the body.

Interaction curves

$$\sigma = \begin{cases} \sigma_0, & E\varepsilon > \sigma_0 \\ E\varepsilon, & -\sigma_0 < E\varepsilon < \sigma_0 \\ -\sigma_0, & E\varepsilon < -\sigma_0 \end{cases}$$

Main formulae resulting from the equivalence theorem:

$$\boxed{\begin{cases} \int \sigma_x \mathrm{dA} = N, & \int \left(\tau_{xz} y - \tau_{xy} z\right) \mathrm{dA} = M_x \\ \int \tau_{xy} \mathrm{dA} = Q_y, & \int \sigma_x z \mathrm{dA} = M_y, \\ \int \tau_{xz} \mathrm{dA} = Q_z, & \int \sigma_x y \mathrm{dA} = M_z. \end{cases}}$$

We introduce dimensionless quantities:

$$s \equiv \frac{\sigma}{\sigma_0}, \quad n = \frac{N}{\overline{\overline{N}}} = \frac{N}{\overline{\overline{\overline{N}}}} = \frac{1}{A}\int s \mathrm{dA}$$

$$\overline{m}_y = \frac{M_y}{\overline{\overline{M}}_y}, \quad \overline{m}_z = \frac{M_z}{\overline{\overline{M}}_z} \quad \text{and} \quad \overline{\overline{m}}_y = \frac{M_y}{\overline{\overline{\overline{M}}}_y}, \quad \overline{\overline{m}}_z = \frac{M_z}{\overline{\overline{\overline{M}}}_z}$$

Elastic range

The normal stress is given by formula of a plane:

$$\sigma_x = \frac{N}{F} + \frac{M_y}{I_y} z - \frac{M_z}{I_z} y \, .$$

In practice, position of the neutral axis suffices to determine point(s) of the yielding onset.
With dimensionless parameters, we have:

$$s \equiv s(y,z) = n + \overline{m}_y \frac{z}{z_{\max}} - \overline{m}_z \frac{y}{y_{\max}} \, .$$

The neutral axis equation:

$$s(y,z) = 0 \quad \rightarrow \quad n + \overline{m}_y \frac{z}{z_{\max}} - \overline{m}_z \frac{y}{y_{\max}} = 0$$

is an equation of straight line. Extreme values with yielding will be attained at:

$$s(y_p, z_p) = \pm 1 \quad \rightarrow \quad n + \overline{m}_y \frac{z_p}{z_{\max}} - \overline{m}_z \frac{y_p}{y_{\max}} = \pm 1 .$$

Biaxial bending case

For $n = 0$, the neutral axis is:

$$\overline{m}_y \frac{z}{z_{\max}} - \overline{m}_z \frac{y}{y_{\max}} = 0$$

and relationship between bending moments is linear:

$$\overline{m}_y \frac{z_p}{z_{\max}} - \overline{m}_z \frac{y_p}{y_{\max}} = \pm 1 .$$

Bending with axial force

$$s = n + \overline{m}_y \frac{z}{z_{\max}} ,$$

and in limit elastic state:

$$n + \overline{m}_y \frac{z_p}{z_{\max}} = \pm 1 ,$$

where z_p is neutral axis position.

Extreme value of bending moment can be found from a formula:

$$\sigma_x = a + bz \quad \Rightarrow \quad M = \iint_F \sigma_x z dF = a \iint_F z dF + b \iint_F z^2 dF = aS_y + bI_y = bI_y ,$$

it means for simultaneous yielding of both extreme fibers, and the neutral axis position:

$$z_0 = \frac{z_g + z_d}{2} .$$

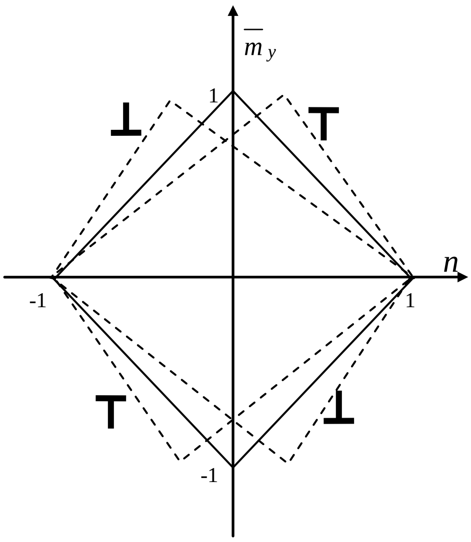


Fig. 3.3 Interaction curves of $n - \overline{m}$

Elastic-plastic range

In the plastic state the stress distribution is partially rectangular

$$s = \begin{cases} \pm 1, & \text{dla } z > z_0 \\ \mp 1, & \text{dla } z < z_0 \end{cases} ,$$

and relations between cross-section forces are non-linear:

$$n = \iint_F s \mathrm{dA} = \pm (A_1 - A_2),$$
$$m = \iint_F sz \mathrm{dA} = \pm (S_{y1} - S_{y2}) .$$

The typical curves are shown in the figure below.

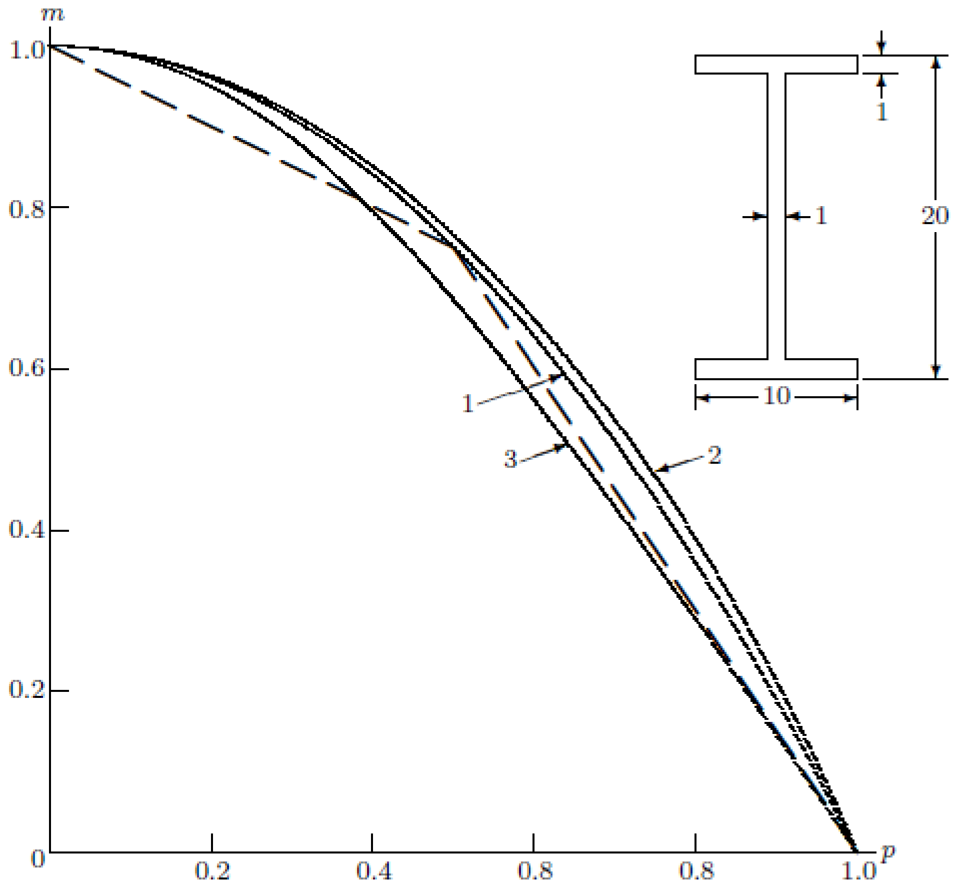


Fig. 3.4 *n-m* curves for rectangle (1), circle (2) and I-beam (3)

Theorems of limit analysis

Introduction

Analyzing a structure, we seek:

– limit value of the load causing the onset of the plastic mechanism
– stress field corresponding to the equilibrium state and the static boundary conditions
– appropriate displacement field or the rate of this field which fulfills kinematic boundary conditions.

The exact solution fulfills the principle of virtual work.

The work (the power) of stress on the displacements (or their rates) is equal to the work (the power) of external forces on the displacements (or their rates).

$$\iint\limits_{A_T} p_j \dot{u}_j \mathrm{dA} + \iiint\limits_{V} F_j \dot{u}_j \mathrm{dV} = \iiint\limits_{V} \sigma_{ij} \dot{\varepsilon}_{ij} \mathrm{dV}$$

Neglecting mass forces, we can write the equation by a coefficient of external forces. The coefficient has an exact value for true forces.

$$m \iint\limits_{A_T} p_{j0} \dot{u}_j \mathrm{dA} = \iiint\limits_{V} \sigma_{ij} \dot{\varepsilon}_{ij} \mathrm{dV}$$

Lemma

If the limit plastic state is reached and displacements increase under constant load, the stress remains constant and only plastic (not elastic) strain increases.

Proof: („rate" form of principle of virtual work)

$$\int_{A_\sigma} \dot{q}_i^l \dot{u}_i^l dA + \int_{A_u} \dot{q}_i^l \dot{u}_i^l dA + \int_V \dot{F}_i^l \dot{u}_i^l dV = \int_V \dot{\sigma}_{ij}^l \dot{\varepsilon}_{ij}^l dV$$

(index *l* means limit state)

For the limit load, left side of the equation vanishes, from the definition:

– $\dot{F}_i^l = 0$ in volume *V*,

– $\dot{q}_i^l = 0$ at A_σ,

– $\dot{u}_i^l = 0$ at A_u.

We decompose the strain rate into elastic and plastic parts:

$$\int_V \dot{\sigma}_{ij}^l \dot{\varepsilon}_{ij}^l dV = \int_V \dot{\sigma}_{ij}^l \left(\dot{\varepsilon}_{ij}^{e_l} + \dot{\varepsilon}_{ij}^{p_l} \right) dV = 0$$

From the associated flow rule for perfectly plastic material follows that the vector $\dot{\sigma}_{ij}$ is tangent to the flow surface, if the plastic strain appears. So:

$$\int_V \dot{\sigma}_{ij}^l \dot{\varepsilon}_{ij}^{e_l} dV = 0,$$

and as consequence, the stress is constant and the rate of elastic strain is zero. For the limit load only the plastic strain exists. The elastic properties of material are not important.

In this way, the model of perfectly elastic-plastic material is equivalent to the model of perfectly plastic material.

Statically admissible solutions

The stress field is *statically admissible*, if the following conditions are fulfilled:

– the equations of internal equilibrium

– static boundary conditions

– the yield criterion in the form of weak inequality (and in particular, the stress does not exceed the plasticity limit).

In such case, the multiplier will be different:

$$m_s \iint_{A_T} p_{j0} \dot{u}_j \mathrm{dA} = \iiint_V \sigma^*_{ij} \, \dot{\varepsilon}_{ij} \mathrm{dV}$$

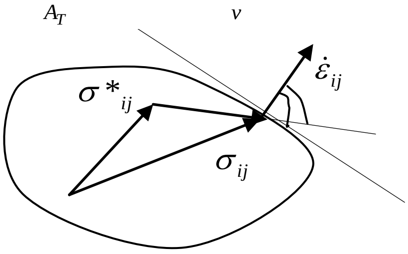


Fig. 3.5 Statically admissible stress field and the sign of the integral

Subtracting the above equations, we get:

$$(m - m_s) \iint_{A_T} p_{j0} \dot{u}_j \mathrm{dA} = \iiint_V \left(\sigma_{ij} - \sigma^*_{ij} \right) \dot{\varepsilon}_{ij} \mathrm{dV} .$$

From the Drucker's stability postulate follows that the sign of the integral on the right side is nonnegative and:

$$m_s \le m .$$

Lower-bound theorem:

The structure does not undergo destruction, or, at the most is in the state of limit equilibrium, if the statically admissible state of stress balances the actual loading.
In other words, the structure does not collapse if the external loading can be balanced by the statically admissible state of stress. The real bearing capacity is at least as the balanced load and it is the lower bound estimation.

Kinematically admissible solutions

The field of displacement rates is kinematically admissible if the following conditions are fulfilled:

– kinematic boundary conditions and compatibility equations
– the condition of nonnegative work (power) of external forces:

$$D_z = m\iint_{A_\sigma} p_{j0}\dot{u}_j^k > 0$$

Applying the principle of virtual work to an arbitrary kinematically admissible field of displacements, we have:

$$m_k\iint_{A_T} p_{j0}\dot{u}_j^k \mathrm{dA} = \iiint_{v} \sigma_{ij}^k\dot{\varepsilon}_{ij}^k \mathrm{dV}\,.$$

Subtracting the above equation from this of "real" state, we get:

$$(m_k - m)\iint_{A_T} p_{j0}\dot{u}_j^k \mathrm{dA} = \iiint_{\mathrm{V}} \left(\sigma_{ij}^k - \sigma_{ij}\right)\dot{\varepsilon}_{ij}^k \mathrm{dV}\,.$$

From the Drucker's stability postulate follows that the integral on the right side is nonnegative. The integral on the left is positive (positive work of internal forces) and the multiplier of kinematically admissible displacement field is not less than real (exact) value:

$$m_k \geq m\,.$$

Upper-bound theorem:

The structure collapses (becomes a mechanism) or, at least is in limit equilibrium state if for kinematically admissible field of displacements the total work (power) of external forces is not less than the work (power) of internal forces.
In other words, if the structure collapses under external load, its bearing capacity is less or equal to the applied load (upper bound estimation).

Approximate and exact solutions

Comparing the theorems, an assessment is valid:

$$m_s \leq m \leq m_k\,.$$

If the statically admissible state of stress is associated at the same time with kinematically admissible field of displacements, the solution is exact to the real value of limit capacity. Such a solution is called a complete solution.
The theorems are very attractive because, in most cases, the upper and lower estimations can be found very easy. Many satisfactory engineering solutions were found for bar structures, plates and soil.
It is shown as a rule, plausible velocity fields are easier to guess than stress fields, and therefore in many cases only upper estimates are available. Of particular importance are velocity fields called mechanisms, in which the deformation is concentrated at points, lines, or planes, with the remaining parts of the system moving as rigid bodies.

The use of mechanisms for estimating collapse loads antedates the development of plasticity theory. Examples include Coulomb's (1773) method of slip planes for studying the collapse strength of soil, the plastic-hinge mechanism due to Kazinczy (1914) for steel frames, and the yield-line theory of Johansen (1932) for reinforced-concrete slabs, later extended to plates in general.

Limit analysis of trusses

A truss member will be said to fail if it can undergo significant lengthening or shortening with no significant change in the bar force. Failure in this sense can result from yielding, if the material is perfectly plastic or nearly so, or, in the case of a compression member, from buckling. The bar force in a failed member is determined by the failure criterion, not by equilibrium. It can be presumed as known if the properties of the bar are known, and the number of unknown bar forces drops by one, as does the indeterminacy number r. The truss therefore becomes unstable if $r+1$ bars fail. In particular, a statically determinate truss collapses as soon as one bar fails. Equating the external work rate to the total internal dissipation leads to an upper bound.

Example

Find the limit bearing capacity of the truss, where $A_1 = 3\,\text{cm}^2$, $A_2 = 2\,\text{cm}^2$, $A_3 = 5\,\text{cm}^2$, $\sigma_0 = 400\text{MPa}$.

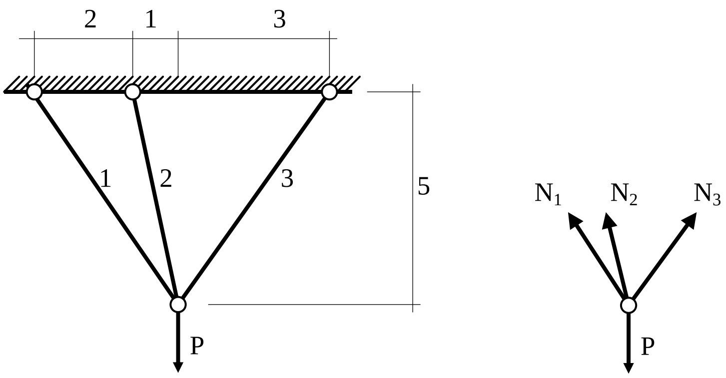


Fig. 3.6 Statically indeterminate truss and the node equilibrium

Static approach

We calculate the bearing capacity of the bars:

$\overline{\overline{N}}_1 = 120\ \text{kN},\ \overline{\overline{N}}_2 = 80\ \text{kN},\ \overline{\overline{N}}_3 = 200\ \text{kN}$

The structure becomes a mechanism if two bars reach yield point. There are three possibilities:

a) $N_1 = \overline{\overline{N}}_1,\quad N_2 = \overline{\overline{N}}_2$

from the sum of projections, we have:

$$\sum X = 0:\quad 120\frac{3}{\sqrt{34}} + 80\frac{1}{\sqrt{26}} = N_3\frac{3}{\sqrt{34}} \quad \rightarrow \quad N_3 = 150.5\ \text{kN}$$

$$\sum Y = 0:\quad P = 120\frac{5}{\sqrt{34}} + 80\frac{5}{\sqrt{26}} + 150.5\frac{5}{\sqrt{34}} = 310.4\ \text{kN}$$

we verify $\sigma_3 = {}^{N_3}\!/_{A_3} = 301\,\text{MPa} < \sigma_0$ (statically admissible scheme)

b) $N_1 = \overline{\overline{N}}_1,\quad N_3 = \overline{\overline{N}}_3$

from the projections sum on x axis, we have:

$$\sum X = 0: \quad \to \quad 120\frac{3}{\sqrt{34}} + N_2\frac{1}{\sqrt{26}} - 200\frac{3}{\sqrt{34}} = 0 \quad \to \quad N_2 = 209.9\,\text{kN} > \overline{\overline{N}}_2$$

so, the scheme is not statically admissible

c) $N_2 = \overline{\overline{N}}_2, \quad N_3 = \overline{\overline{N}}_3$

from the projection on horizontal axis we have:

$$\sum X = 0: \quad N_1\frac{3}{\sqrt{34}} + 80\frac{1}{\sqrt{26}} - 200\frac{3}{\sqrt{34}} = 0 \quad \to \quad N_1 = 169.5\,\text{kN} > \overline{\overline{N}}_1$$

and the scheme is not statically admissible.

Actually, we have only one scheme statically admissible and the lower bound estimation is 310.4 kN.

Kinematic approach

As before, there are three possibilities of changing the structure into mechanism. We verify only one of them that corresponds to yielding of bars *1* and *2*. The system has an instant center of rotation at the end of the bar no 3.

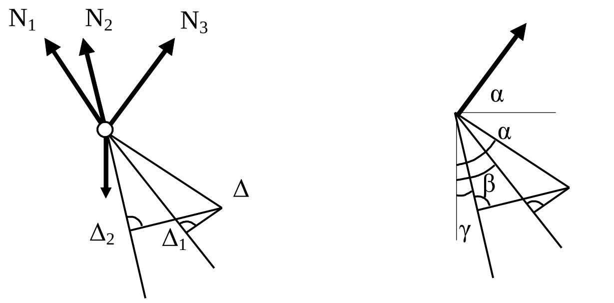


Fig. 3.7 Kinematic scheme of the mechanism

Comparing the work of external and internal forces, we have:

$$\mathbf{P}\cdot\mathbf{\Delta} = \overline{\overline{\mathbf{N}}}_1\cdot\mathbf{\Delta}_1 + \overline{\overline{\mathbf{N}}}_2\cdot\mathbf{\Delta}_2 \quad \to \quad P\Delta\frac{3}{\sqrt{34}} = 120\Delta_1 + 80\Delta_2$$

and from geometrical relations follows:

$$\alpha = \arccos\frac{3}{\sqrt{34}} = 59.04^0, \quad \beta = \arccos\frac{5}{\sqrt{34}} = 30.96^0, \quad \gamma = \arccos\frac{5}{\sqrt{26}} = 11.31^0$$

$$\Delta_1 = \Delta\cos(\alpha-\beta) = 0.8823\Delta, \quad \Delta_2 = \Delta\cos(\alpha-\gamma) = 0.6726\Delta$$

so:

$P = 310.4$ kN,

and the result is identical with the static approach result. We have exact solution for the model.

Limit analysis of beams

Any transversely loaded beam is statically indeterminate in the sense that the stress field cannot be deduced from the loading independently of unknown properties. It is conventional, however, to call a beam statically determinate or indeterminate if it is externally determinate or indeterminate. The number of equilibrium equations available is three for plane bending. Any hinge, whether frictionless or a plastic hinge, provides an additional equilibrium equation: at a frictionless hinge, $M = 0$, since such a hinge cannot transmit moment, while at a plastic hinge $M = \mp\overline{\overline{M}}$.

The beam collapses when the number of hinges reduces the number of indeterminacy to -1. If the beam is statically determinate one hinge is sufficient. A plastic hinge may form at any point of the beam at which the condition $|M| = \overline{\overline{M}}$ is possible, that

is, in the interior of a span, at a fixed-end, or at an intermediate support. A collapse mechanism is admissible if it does not violate any support condition (unless relaxed by the formation of a plastic hinge) and if it produces positive external work rate.
A hinge that has rotated by an angle Θ can be thought of as the limit of a small segment of length Δx. The total internal dissipation in the hinge is $\theta\overline{\overline{M}}$.
A moment distribution is statically and plastically admissible if it is in equilibrium with the applied load, is consistent with all force and moment end conditions and frictionless hinge conditions (if any), and is such that $|M| \le \overline{\overline{M}}$ everywhere. In the moment distribution at collapse, the points where $M = \pm\overline{\overline{M}}$ are precisely the ones where plastic hinges form.

Example
Find limit bearing capacity of the beam.

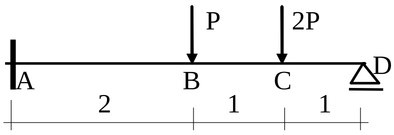


Fig. 3.8 Beam scheme

Static approach
We apply the method of consecutive plastic hinges. The plastic hinges carry on the limit plastic bending moments, $\overline{\overline{M}}$, the direction of which corresponds with the stretched fibers. To determine the section of the first hinge we have to have the diagram of bending moments. Due to its form of linear segments we consider three possible sections only.
a) For the hinge in the section A, we get the statically determined beam

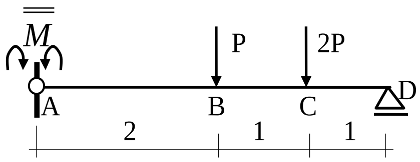


Fig. 3.9 Scheme with the hinge at A

$$R_A = P + \frac{\overline{\overline{M}}}{4}, \quad R_D = 2P - \frac{\overline{\overline{M}}}{4}$$

$$M_B = 2P - \frac{\overline{\overline{M}}}{2}, \quad M_C = 2P - \frac{\overline{\overline{M}}}{4}$$

Because of $M_B < M_C$, we assume the next plastic hinge in the section C. The limit load and the bending moment in the section B will be:

$$\overline{\overline{P}} = \tfrac{5}{8}\overline{\overline{M}}, \quad M_B = \tfrac{3}{4}\overline{\overline{M}} < \overline{\overline{M}}\,.$$

The scheme is statically admissible because the bending moment M_B is less than plastic limit moment.
b) For the first plastic hinge in the section B, we have:

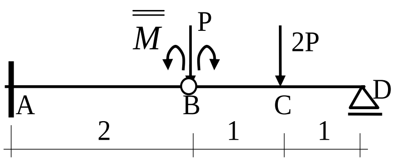


Fig. 3.10 Scheme with the hinge at B

$$R_D = P + \overline{\overline{M}}/2,\ R_B = P - \overline{\overline{M}}/2,\ M_A = 2\overline{\overline{M}} - 4P,\ M_C = P + \overline{\overline{M}}/2 .$$

For two possibilities for second hinge we get, at the section A:

$$\overline{\overline{P}} = \tfrac{1}{4}\overline{\overline{M}}, \quad M_C = \tfrac{3}{4}\overline{\overline{M}} < \overline{\overline{M}} \text{ (admissible)}$$

and in the section C:

$$\overline{\overline{P}} = \overline{\overline{M}}/2, \text{ and } M_A = 0 \text{ (admissible).}$$

c) For the first hinge in the section C, we have:

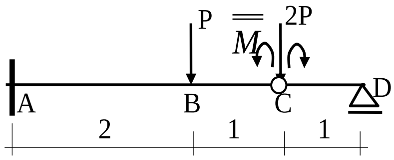


Fig. 3.11 Scheme with the hinge at C

$$R_D = \overline{\overline{M}},\ R_C = -\overline{\overline{M}},\ M_B = 2\overline{\overline{M}} - 2P,\ M_A = -4\overline{\overline{M}} + 8P .$$

The next hinge in the section A, we have: $\overline{\overline{P}} = \frac{5}{8}\overline{\overline{M}}$, $M_B = \frac{3}{4}\overline{\overline{M}} < \overline{\overline{M}}$,

and for the next hinge in the section B, we get: $\overline{\overline{P}} = \frac{1}{2}\overline{\overline{M}}$, $M_A = 0 < \overline{\overline{M}}$.

From these lower bound estimations we choose the biggest value. So, the limit plastic bearing capacity of the beam is equal to the maximum of lower bound estimations.

$$\overline{\overline{P}} = \max\left(\tfrac{5}{8}\overline{\overline{M}}, \tfrac{1}{4}\overline{\overline{M}}, \tfrac{1}{2}\overline{\overline{M}}\right) = \tfrac{5}{8}\overline{\overline{M}} .$$

Kinematical approach

The most probable sections of plastic hinges are the sections at the intervals ends. To obtain the kinematic mechanism with one degree of freedom (DOF) we need two plastic hinges. Only three sections are involved and, consequently, three kinematic schemes.

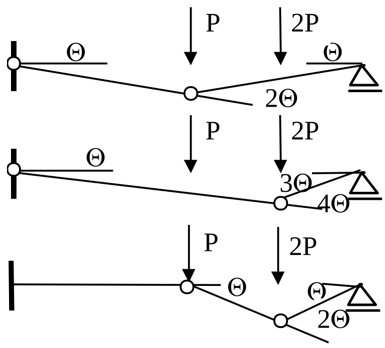


Fig. 3.12 Kinematic schemes

Comparing the external and internal work, we get:

$$2\Theta P + 2P\Theta = 3\overline{\overline{M}}\Theta \quad \Rightarrow \quad \overline{\overline{P}}_1 = 0.75\overline{\overline{M}}$$

$$2\Theta P + 2P3\Theta = 5\overline{\overline{M}}\Theta \quad \Rightarrow \quad \overline{\overline{P}}_2 = 0.625\overline{\overline{M}}$$

$$2P\Theta = 3\overline{\overline{M}}\Theta \quad \Rightarrow \quad \overline{\overline{P}}_3 = 1.5\overline{\overline{M}}$$

The kinematic approach is the upper bound estimation, so we choose the smallest value of estimation (the beam collapses under the force $\overline{\overline{P}}_1, \overline{\overline{P}}_2$ as well as $\overline{\overline{P}}_3$), so the best value is $\overline{\overline{P}} = 0.625\overline{\overline{M}}$. The same result we got from the static approach, so, the result is exact.

Example

Find the limit bearing capacity of the statically undetermined beam.

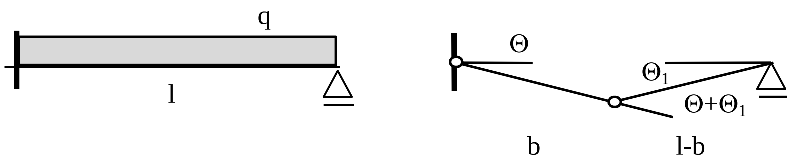


Fig. 3.13 Beam and the kinematic scheme of collapse

Similarly as before, two plastic hinges will be necessary to create the kinematical mechanism with one DOF. One hinge will be at the fixed end but the position of the second hinge is unknown. We assume hypothetically the second hinge in the middle of the span, $b=\frac{1}{2}l, \Theta_1=\Theta$. We get:

$$2\int_0^{0.5l} q\Theta x dx = 3\Theta\overline{\overline{M}} \quad \rightarrow \quad \bar{\bar{q}} = 12\frac{\overline{\overline{M}}}{l^2}.$$

From the static approach the hypothesis gives us:

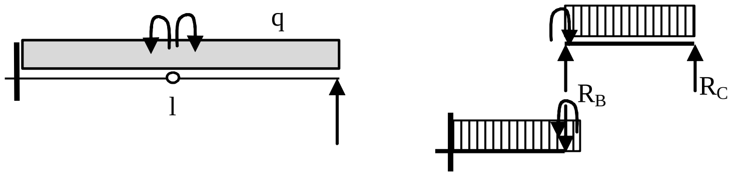


Fig. 3.14 Static scheme of the beam

The reaction in the upper beam is:

$$R_B = \frac{ql}{4} - \frac{2\overline{\overline{M}}}{l}$$

and the bending moment at the fixed end of the lower beam is

$$M_u = \frac{ql^2}{4} - 2\overline{\overline{M}}$$

and with the second hinge at the fixed end, we have:

$$M_u = \overline{\overline{M}} \quad \rightarrow \quad \bar{\bar{q}} = 12\frac{M}{l^2},$$

so, the same value as from the kinematic approach. Seemingly, the solution looks to be exact, but it is not the case. When we calculate the reaction for the limit value found, we get:

$$R_B = 3\frac{\overline{\overline{M}}}{l} - 2\frac{\overline{\overline{M}}}{l} = \frac{\overline{\overline{M}}}{l} > 0,$$

and this signifies that on the right of the hinge the shear force changes the sign and has a zero-value point:

$$R_B - \frac{\bar{\bar{q}}l}{2} = \frac{\overline{\overline{M}}}{l} - 6\frac{\overline{\overline{M}}}{l} = -5\frac{\overline{\overline{M}}}{l}.$$

It means that there is extreme bending moment in the right span of the beam. This extreme value must be greater than the value at the hinge due to convexity of the bending moment diagram. The value exceeds the limit bending moment and the scheme is statically not admissible.
Let's change the sequence of the hinges. For the plastic hinge at the fixed end, we have:

$$R_A = \frac{ql}{2} + \frac{\overline{\overline{M}}}{l}$$

and the shear force in the span:

$$Q(x) = R_A - qx\,.$$

From the condition of zero-value of the shear force we get the position of the second hinge:

$$Q(x) = 0 \quad \rightarrow \quad x_{extr} = \frac{R_A}{q}$$

and extreme value of the moment will be:

$$M(x_{extr}) = R_A x_{extr} - \overline{\overline{M}} - \frac{q x_{extr}^2}{2} = \frac{R_A^2}{2q} - \overline{\overline{M}}$$

Assuming the second hinge created we find the limit load of the beam:

$$M(x_{extr}) = \overline{\overline{M}} \quad \rightarrow \quad \frac{\overline{\overline{q}}^2 l^2}{4} - 3\overline{\overline{q}}\,\overline{\overline{M}} + \frac{\overline{\overline{M}}^2}{l^2} = 0$$

We change the equation introducing a new variable:

$$\xi = \frac{\overline{\overline{q}} l^2}{\overline{\overline{M}}} \quad \rightarrow \quad \tfrac{1}{4}\xi^2 - 3\xi + 1 = 0 \quad \rightarrow \quad \xi_1 = 0.343,\ \xi_2 = 11.66\,.$$

Because the static approach gives the lower bound estimation we take the second core and the corresponding limit load value:

$$\overline{\overline{q}} = 11.66\frac{\overline{\overline{M}}}{l^2}\,.$$

Similarly, we find from the kinematical scheme the exact value of load capacity. We find the position of the hinge at the span from the principle of virtual work:

$$\int_0^b q\Theta x\,dx + \int_0^{l-b} q\Theta_1 x_1\,dx_1 = \overline{\overline{M}}\left(2\Theta + \Theta_1\right)$$

and, from the figure is:

$$\Theta_1 = \frac{b}{l-b}\Theta,$$

we have:

$$q = \frac{2\overline{\overline{M}}}{bl}\,\frac{2l-b}{l-b}\,.$$

We seek the extreme (minimal) value of load capacity:

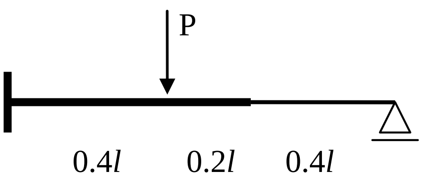
$$\min \overline{\overline{q}} \quad \Rightarrow \quad \frac{\partial q}{\partial b} = 0 \quad \Rightarrow \quad b^2 - 4bl + 2l^2 = 0 \quad \Rightarrow \quad b = l\left(2-\sqrt{2}\right) = 0.59l\,,$$

and finally: $\overline{\overline{q}} = \dfrac{4}{6-4\sqrt{2}}\dfrac{\overline{\overline{M}}}{l^2} = 11.66\dfrac{\overline{\overline{M}}}{l^2}$

Both solutions are identical.

Example

Find the limit load capacity of the beam with variable cross-section capacity: $2\overline{\overline{M}}$ from the left and $\overline{\overline{M}}$ from the right.



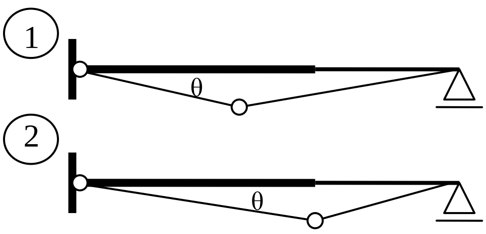


Fig. 3.15 Beam with variable stiffness and collapse schemes

For the kinematical schemes we have:

1) $\Theta_1 = \frac{2}{3}\Theta$, $\quad 2\overline{\overline{M}}(\Theta + \frac{5}{3}\Theta) = P\Theta 0.4l \quad \rightarrow \quad \overline{\overline{P}} = 13.33\frac{\overline{\overline{M}}}{l}$

2) $\Theta_1 = 1{,}5\Theta$, $\quad 2\overline{\overline{M}}\Theta + 2.5\overline{\overline{M}}\Theta = P\Theta 0.4l \quad \rightarrow \quad \overline{\overline{P}} = 11.25\frac{\overline{\overline{M}}}{l}$

The lower value is the solution from the kinematic approach.

Problem

Using static and kinematic approach, determine the plastic limit load for the beam shown in the figure below and $\sigma_0 = 320$ [MPa].

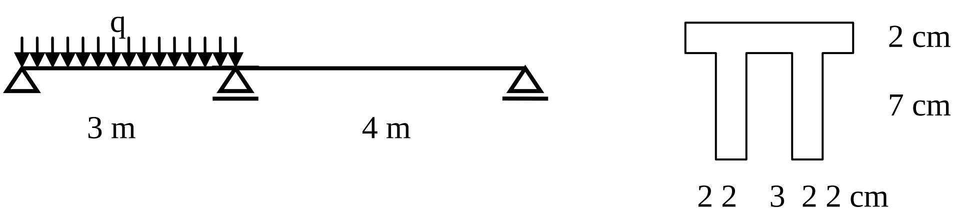


Fig. 3.16 Beam and the cross-section

Solution:

1. The limit bending moment of the cross-section:

– cross-section area A = 50 cm^2
– gravity center: C(5.5, 5.48)
– principal inertia moment J_y = 371.15 cm^4
– cross-section elastic factor W_{el} = 67.73 cm^3
– position of neutral axis at limit bending moment: y_0 = 0.7696 cm (upward from gravity center)
– cross-section plastic factor W_{pl} = 117.75 cm^3
– factors' relation $n = W_{pl}/W_{el}$ = 1.74
– limit bending moment $M_{pl} = 320 \cdot 10^6 \cdot 117.75 \cdot 10^{-6} = 37.68$ kNm

2. Static approach:

– we introduce one plastic hinge which suffices for the beam to evolve into statically determined

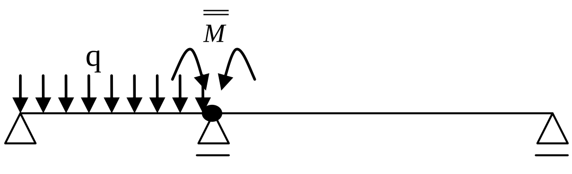


Fig. 3.17 Beam with the hinge

– (two parts of the beam work independently), for the left beam we have:

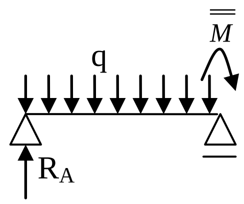


Fig. 3.18 Left part of the beam

$$R_A = \frac{4.5q - \overline{\overline{M}}}{3} = 1.5q - \tfrac{1}{3}\overline{\overline{M}}$$

– we seek the position of zero transverse force:

$$Q(x) = R_A - qx = 1.5q - \tfrac{1}{3}\overline{\overline{M}} - qx = 0 \quad \rightarrow \quad x_p = 1.5 - \frac{\overline{\overline{M}}}{3q}$$

– the bending moment at this point is:

$$M(x_p) = R_A x_p - \frac{q}{2}x_p^2 = \ldots = 1.125q - 0.5\overline{\overline{M}} + \frac{\overline{\overline{M}}^2}{18q}$$

– we assume the next plastic hinge at the point

$$M(x_p) = \overline{\overline{M}} \quad \rightarrow \quad 1.125q^2 - 1.5\overline{\overline{M}}q + \tfrac{1}{18}\overline{\overline{M}}^2 = 0 \quad \rightarrow \quad \overline{\overline{q}} = \frac{1.5+\sqrt{2}}{2\cdot 1.125}\overline{\overline{M}} = 1.295\overline{\overline{M}}$$

– the plastic limit load is:

$$\overline{\overline{q}} = 1.295\overline{\overline{M}} = 48.8\ \mathrm{kN/m}$$

3. Kinematic approach

– we introduce two plastic hinges to transform the beam into the mechanism with one degree of freedom

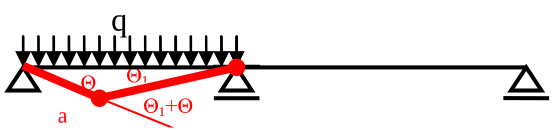


Fig. 3.19 Kinematic scheme

– from geometric relations, we have:

$$\theta a = \theta_1(3-a) \quad \rightarrow \quad \theta_1 = \theta\frac{a}{3-a}$$

– the work of internal forces (always positive)

$$L_{in} = \overline{\overline{M}}(\theta+\theta_1) + \overline{\overline{M}}\theta_1 = \ldots = \overline{\overline{M}}\theta\frac{3+a}{3-a}$$

– the work of external forces

$$L_{ex} = \int_0^a q\theta x dx + \int_0^{3-a} q\theta_1 x dx = \ldots = 1.5qa\theta$$

– the bearing capacity of the scheme

$$L_{\text{int}} = L_{ext} \quad \rightarrow \quad q = \frac{2}{3}\frac{3+a}{(3-a)a}\overline{\overline{M}}$$

– we seek the extreme value of the load

$$\frac{\partial q}{\partial a} = 0 \quad \rightarrow \quad 3a - a^2 - (3+a)(3-2a) = 0 \quad \rightarrow \quad a = 3(\sqrt{2}-1) = 1.243\ \mathrm{m}$$

– finally, the plastic limit load is:

$$\overline{\overline{q}} = \frac{2}{3}\frac{3+3(\sqrt{2}-1)}{(3-3\sqrt{2}+3)3(\sqrt{2}-1)}\overline{\overline{M}} = \ldots = \frac{2\sqrt{2}}{9(3\sqrt{2}-4)}\overline{\overline{M}} = 1.295\overline{\overline{M}} = 48.8 \text{ kN/m}$$

(the same value as in the static approach, it means the exact solution)

Problem

Using static and kinematic approach, determine the plastic limit load for the beam shown in the figure below and $\sigma_0 = 330$ [MPa].

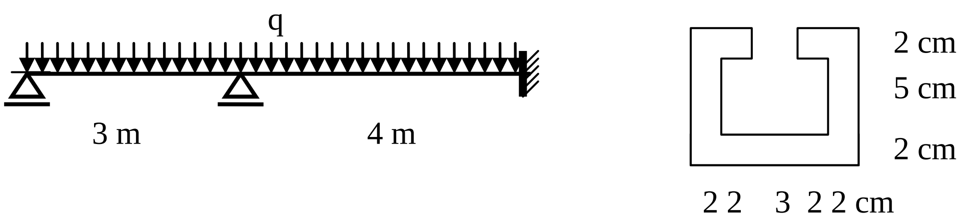


Fig. 3.20 Beam and the cross-section

Solution

1. The limit bending moment of the cross-section:

– cross-section area A = 58 cm^2
– gravity center: C(5.5, 4.14)
– principal inertia moment J_y = 512.23 cm^4
– cross-section elastic factor W_{el} = 105.35 cm^3
– position of neutral axis at limit bending moment: y_0 = 0.39 cm (downward from gravity center)
– cross-section plastic factor W_{pl} = 155.75 cm^3
– factors' relation $n = W_{pl}/W_{el}$ = 1.478
– limit bending moment $M_{pl} = 330 \cdot 10^6 \cdot 155.75 \cdot 10^{-6}$ = 51.40 kNm

2. Static approach:

left part of the beam:

– we introduce one plastic hinge which suffices for the beam to evolve into statically determined

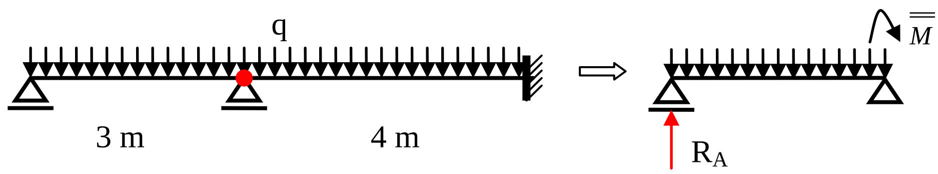


Fig. 3.21 Beam with the plastic hinge

$$R_A = \frac{4.5q - \overline{\overline{M}}}{3} = 1.5q - \tfrac{1}{3}\overline{\overline{M}}$$

– we seek the position of zero transverse force:

$$Q(x) = R_A - qx = 1.5q - \tfrac{1}{3}\overline{\overline{M}} - qx = 0 \quad \rightarrow \quad x_p = 1.5 - \frac{\overline{\overline{M}}}{3q}$$

– the bending moment at this point is:

$$M(x_p) = R_A x_p - \frac{q}{2}x_p^2 = \ldots = 1.125q - 0.5\overline{\overline{M}} + \frac{\overline{\overline{M}}^2}{18q}$$

– we assume the next plastic hinge at the point

$$M(x_p)=\overline{\overline{M}} \quad \rightarrow \quad 1.125q^2-1.5\overline{\overline{M}}q+\tfrac{1}{18}\overline{\overline{M}}^2=0 \quad \rightarrow \quad \overline{\overline{q}}=\frac{1.5+\sqrt{2}}{2\cdot 1.125}\overline{\overline{M}}=1.295\overline{\overline{M}}$$

– the plastic limit load is:

$$\overline{\overline{q}}=1.295\overline{\overline{M}}=66.56\ \mathrm{kN/m}$$

right part of the beam:

– we introduce two plastic hinges which suffice for the part of the beam to become statically determined

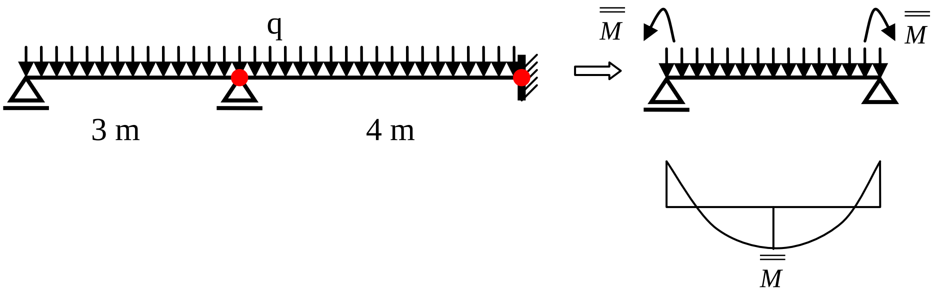


Fig. 3.22 Beam with two plastic hinges

– we assume the third plastic hinge at the middle of the span:

$$\overline{\overline{M}}-\frac{ql^2}{8}=-\overline{\overline{M}} \quad \rightarrow \quad \frac{ql^2}{8}=2\overline{\overline{M}} \quad \rightarrow \quad \overline{\overline{q}}=\frac{16}{l^2}\overline{\overline{M}}=\overline{\overline{M}}=51.40\ \mathrm{kN/m}$$

– the value for the second scheme is less then prior value, so it is evident that the first scheme is statically inacceptable (for this limit load the bending moment at right span will exceed admissible value), and the answer is:

$$\overline{\overline{q}}=51.40\ \mathrm{kN/m}$$

3. Kinematic approach

– we introduce one degree of freedom mechanism at left and right parts of the beam:

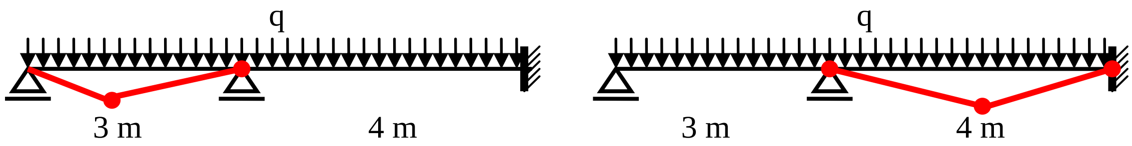


Fig. 3.23 Kinematic scheme

left part of the beam:

– from geometrical relations (the plastic hinge at *a* from the roller), we have:

$$\theta a=\theta_1(3-a) \quad \rightarrow \quad \theta_1=\theta\frac{a}{3-a}$$

– the work of internal forces (always positive)

$$L_{in}=\overline{\overline{M}}(\theta+\theta_1)+\overline{\overline{M}}\theta_1=\ldots=\overline{\overline{M}}\theta\frac{3+a}{3-a}$$

– the work of external forces

$$L_{ex}=\int_0^a q\theta x dx+\int_0^{3-a} q\theta_1 x dx=\ldots=1.5qa$$

– the bearing capacity of the scheme

$$L_{\mathrm{int}}=L_{ext} \quad \rightarrow \quad q=\frac{2}{3}\frac{3+a}{(3-a)a}\overline{\overline{M}}$$

– we seek the extreme value of the load

$$\frac{\partial q}{\partial a}=0 \quad \rightarrow \quad 3a-a^2-(3+a)(3-2a)=0 \quad \rightarrow \quad a=3(\sqrt{2}-1)=1.243\ \text{m}$$

– finally, the plastic limit load is:

$$\bar{\bar{q}}=\frac{2}{3}\frac{3+3(\sqrt{2}-1)}{(3-3\sqrt{2}+3)3(\sqrt{2}-1)}\overline{\overline{M}}=\ldots=\frac{2\sqrt{2}}{9(3\sqrt{2}-4)}\overline{\overline{M}}=1.295\overline{\overline{M}}=66.56\ \text{kN/m}$$

right part of the beam:

– from the external work compared with the internal one, we have:

$$4\overline{\overline{M}}\theta=2\int_0^2 q\theta x dx \quad \rightarrow \quad \bar{\bar{q}}=\overline{\overline{M}}=51.40\ \text{kN/m}$$

– this value is less then the value in the first scheme, and because we take minimum from both cases, the answer is

$$\bar{\bar{q}}=51.40\ \text{kN/m}$$

Problem

Using static and kinematic approach, determine the plastic limit load for the beam shown in the figure below and $\sigma_0=300$ [MPa].

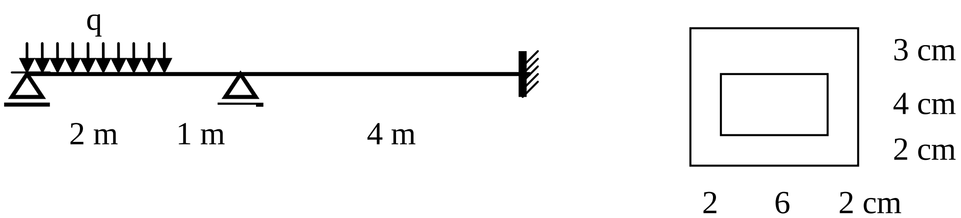


Fig.3.24 Beam with the cross-section

Solution:

1. The limit bending moment of the cross-section:

– cross-section area A = 66 cm^2

– gravity center: C(5, 4.68)

– principal inertia moment J_y = 567.3 cm^4

– cross-section elastic factor W_{el} = 121.75 cm^3

– position of neutral axis at limit bending moment: y_0 = 0.57 cm (upward from gravity center)

– cross-section plastic factor W_{pl} = 174.75 cm^3

– factors' relation $n = W_{pl}/W_{el}$ = 1.44

– limit bending moment $M_{pl} = 300 \cdot 10^6 \cdot 174.75 \cdot 10^{-6}$ = 52.43 kNm

2. Static approach:

– we introduce one plastic hinge which suffices for the beam to evolve into statically determined (two parts of the beam work independently)

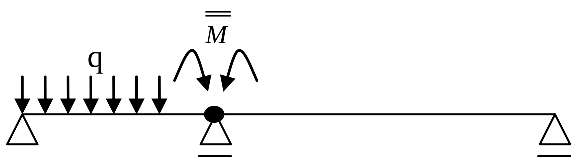


Fig. 3.25 Beam with the hinge

– for the left beam we have:

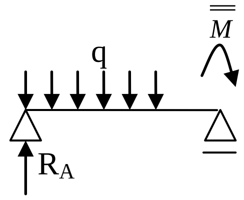


Fig. 3.26 Left part of the beam

$$R_A = \frac{2q2 - \overline{\overline{M}}}{3}$$

– we seek the position of zero transverse force:

$$Q(x) = R_A - qx = 0 \quad \rightarrow \quad x_p = \frac{R_A}{q}$$

– the bending moment at this point is:

$$M(x_p) = R_A x_p - \frac{q}{2} x_p^2 = \ldots = \frac{R_A^2}{2q}$$

– we assume the next plastic hinge at the point

$$M(x_p) = \overline{\overline{M}} \quad \rightarrow \quad \frac{R_A^2}{2q} = \overline{\overline{M}} \quad \rightarrow \quad 16q^2 - 26\overline{\overline{M}}q + \overline{\overline{M}}^2 = 0$$

– we have two roots; but only the positive one makes sense:

$$\overline{\overline{q}} = 1.5856\overline{\overline{M}} \quad \rightarrow \quad R_A = 1.781\overline{\overline{M}} \quad \rightarrow \quad x_p = 1.123\,\mathrm{m}$$

– the plastic limit load is:

$$\overline{\overline{q}} = 1.5856\overline{\overline{M}} = 83.13\,\mathrm{kN/m}$$

3. Kinematic approach

– we introduce the plastic hinges to transform the beam into the mechanism with one degree of freedom

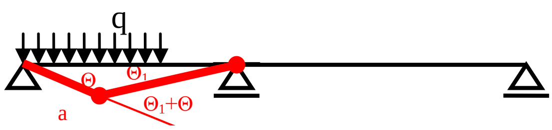


Fig. 3.27 Kinematic scheme

– from geometrical relations, we have:

$$\theta a = \theta_1(3-a) \quad \rightarrow \quad \theta_1 = \theta\frac{a}{3-a}$$

– the work of internal forces (always positive)

$$L_{in} = \overline{\overline{M}}(\theta + \theta_1) + \overline{\overline{M}}\theta_1 = \ldots = \overline{\overline{M}}\theta\frac{3+a}{3-a}$$

– the work of external forces

$$L_{ex} = \int_0^a q\theta x dx + \int_1^{3-a} q\theta_1 x dx = \ldots = 0.5q\theta\left(a^2 + \frac{a}{3-a}\left[(3-a)^2 - 1^2\right]\right)$$

– the bearing capacity of the scheme

$$L_{\text{int}} = L_{ext} \quad \rightarrow \quad q = \frac{2(3+a)}{-3a^2 + 8a}\overline{\overline{M}}$$

– we seek the extreme value of the load

$$\frac{\partial q}{\partial a} = 0 \quad \rightarrow \quad 2(-3a^2 + 8a) - (6 + 2a)(-6a + 8) = 0 \quad \rightarrow \quad a^2 + 6a - 8 = 0 \quad \rightarrow \quad a = 1.123 \text{ m}$$

– finally, the plastic limit load is:

$$\overline{\overline{q}} = 1.586\overline{\overline{M}} = 83.13 \text{ kN/m}$$

(the same value as in the static approach, it means the exact solution)

Problem

Using static and kinematic approach, determine the plastic limit load for the beam shown in the figure below and $\sigma_0 = 250$ [MPa].

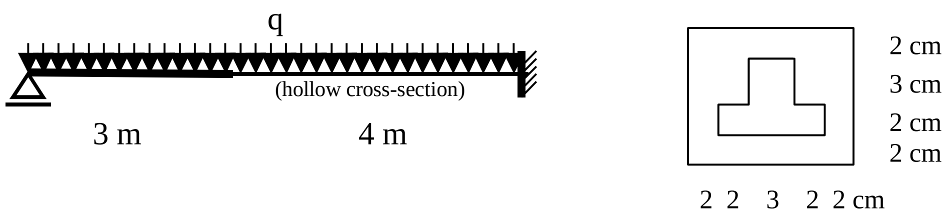


Fig. 3.28 Beam and the cross-section

Solution

1. The limit bending moment of the cross-section:

(solid cross-section, without a hole)

– cross-section area A = 99 cm^2
– gravity center: C(5.5, 4.5)
– principal inertia moment J_y = 668.25 cm^4
– cross-section elastic factor W_{el} = 148.5 cm^3
– position of neutral axis at limit bending moment: y_0 = 0.0 cm (at the gravity center)
– cross-section plastic factor W_{pl} = 222.75 cm^3
– factors' relation $n = W_{pl}/W_{el}$ = 1.5
– limit bending moment $M_{pl} = 250 \cdot 10^6 \cdot 222.75 \cdot 10^{-6}$ = 55.69 kNm

(hollow cross-section)

– cross-section area A = 76 cm^2
– gravity center: C(5.5, 4.66)
– principal inertia moment J_y = 614.44 cm^4
– cross-section elastic factor W_{el} = 131.9 cm^3
– position of neutral axis at limit bending moment: y_0 = 0.34 cm (upward from the gravity center)
– cross-section plastic factor W_{pl} = 190 cm^3
– factors' relation $n = W_{pl}/W_{el}$ = 1.44
– limit bending moment $M_{pl} = 250 \cdot 10^6 \cdot 190 \cdot 10^{-6}$ = 47.5 kNm

2. Static approach:

– we introduce one plastic hinge which suffices for the beam to evolve into statically determined

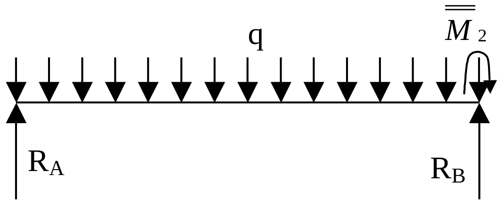


Fig. 3.29 Beam with the hinge

– the reaction is

$$R_A = \frac{\frac{ql^2}{2} - \overline{\overline{M}}_2}{l} = 3.5q - 6.786$$

– the shear force is

$$Q(x) = R_A - qx = 3.5q - qx - 6.786 = 0$$

– extreme value of bending moment occurs at the point of zero shear force

$$x_{\max} = 3.5 - \frac{6.786}{q}$$

– extreme bending moment

$$M_{\max} = R_A x_{\max} - \frac{qx_{\max}^2}{2} = \overline{\overline{M}}_1 = 55.69 \quad \rightarrow \quad 6.125q^2 - 79.19q + 23.02 = 0$$

– so, the limit load is

$$q = 12.63\,\text{kN/m}$$

– we verify the bending moment value at the point of cross-section change

$$x_{\max} = 2.963, \quad R_A = 37.42, \quad M(3) = 37.42 \cdot 3 - 0.5 \cdot 12.63 \cdot 3^2 = 55.4 \quad > \overline{\overline{M}}_2$$

(statically not admissible)

– we assume plastic hinge at the point of cross-section change

$$M(3) = 3R_A - \frac{q}{2}3^2 = 6q - 20.36 = \overline{\overline{M}}_2 = 47.5$$

– so, the limit load is:

$$q = 11.31\,\text{kN/m}$$

3. Kinematical approach:

– we assume the plastic hinge at the point of cross-section change

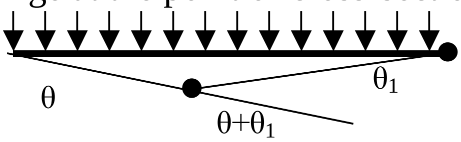


Fig. 3.30 Kinematic scheme

– external work is

$$L_{ex} = 47.5\left(1 + 2 \cdot \frac{3}{4}\right)\theta = 118.75\theta$$

– internal work is

$$L_{in} = \int_0^3 q\theta x dx + \int_0^4 q\frac{3}{4}\theta x dx = 10.5q\theta$$

– comparing, we get the limit load

$$\overline{\overline{q}} = 11.31\,\text{kN/m}$$

(the same value as for the static approach)

Problem

Using static and kinematic approach, determine the plastic limit load for the beam shown in the figure below and $\sigma_0 = 350$ [MPa].

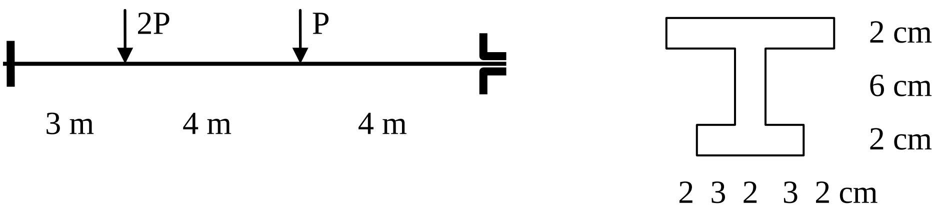


Fig. 3.31 Beam and the cross-section

Solution:

1. The limit bending moment of the cross-section:

– cross-section area A = 52 cm^2
– gravity center: C(6, 5.615)
– principal inertia moment J_y = 669.6 cm^4
– cross-section elastic factor W_{el} = 119.25 cm^3
– position of neutral axis at limit bending moment: y_0 = 1.385 cm (upward from gravity center)
– cross-section plastic factor W_{pl} = 170.0 cm^3
– factors' relation $n = W_{pl}/W_{el} = 1.43$
– limit bending moment $M_{pl} = 350 \cdot 10^6 \cdot 170 \cdot 10^{-6}$ = 59.5 kNm

2. Static approach:

– we introduce two plastic hinges which suffice for the beam to evolve into statically determined

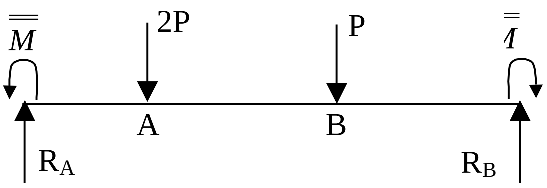


Fig. 3.32 Beam with two hinges

– we calculate the bending moments at the point of forces applied

$$R_A = \frac{2P \cdot 8 + 4P}{11} = 1.82P$$

$$M_A = -\overline{\overline{M}} + 3R_A = -\overline{\overline{M}} + 5.46P\,,\ M_B = -\overline{\overline{M}} + 7R_A - 4 \cdot 2P = -\overline{\overline{M}} + 4.74P\,,\ M_A > M_B$$

$$M_A = \overline{\overline{M}} \quad \rightarrow \quad \overline{\overline{P}} = 0.3667\overline{\overline{M}} \quad \rightarrow \quad M_B = 0.735\overline{\overline{M}} \quad < \overline{\overline{M}} \quad OK$$

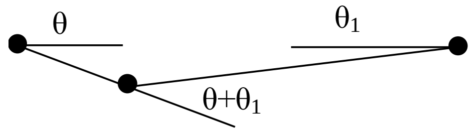
– the limit load is:

$$\overline{\overline{P}} = 0.3667\overline{\overline{M}} = 21.82 \text{ kN}$$

3. Kinematical approach:

we consider two schemes of the kinematical mechanisms

scheme 1



$$\theta_1 = \tfrac{3}{8}\theta$$

Fig. 3.33 Kinematic scheme 1

– external work

$$L_{ex} = \overline{\overline{M}}\theta + \overline{\overline{M}}(\theta+\theta_1) + \overline{\overline{M}}\theta_1 = \tfrac{11}{4}\theta\overline{\overline{M}}$$

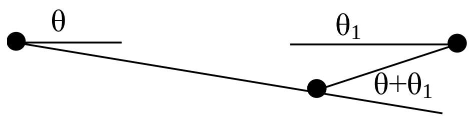
– internal work

$$L_{in} = 2P\theta 3 + P\tfrac{3}{8}\theta 4 = 7.5\theta P$$

– limit load

$$\overline{\overline{P}} = 0.3667\overline{\overline{M}}$$

scheme 2



$$\theta_1 = \tfrac{7}{4}\theta$$

Fig. 3.34 Kinematic scheme 2

– external work

$$L_{ex} = \overline{\overline{M}}\theta + \overline{\overline{M}}(\theta+\theta_1) + \overline{\overline{M}}\theta_1 = 5.5\theta\overline{\overline{M}}$$

– internal work

$$L_{in} = 2P\theta 3 + P\theta 7 = 13\theta P$$

– limit load

$$\overline{\overline{P}} = 0.423\overline{\overline{M}}$$

– we choose

$$\overline{\overline{P}} = \min\left(\overline{\overline{P}}_1, \overline{\overline{P}}_2\right) = 0.3667\overline{\overline{M}} = 21.82\ \text{kN}$$

(the same value as in static approach)

Limit analysis of frames

An assemblage of bars that are joined together rigidly resists the applied loads primarily through bending; axial force and shear are considered secondary effects. Collapse is assumed to occur when sufficient plastic hinges have formed to produce a mechanism. In a multistory frame, collapse may be limited to a single story, and therefore the overall degree of static indeterminacy is not a relevant parameter for the determination of the necessary number of hinges.
A one-story, one-bay frame, see Fig. below, is statically indeterminate of degree three, and the collapse of the frame as a whole indeed requires four hinges, as shown in Fig. (a) and (c).

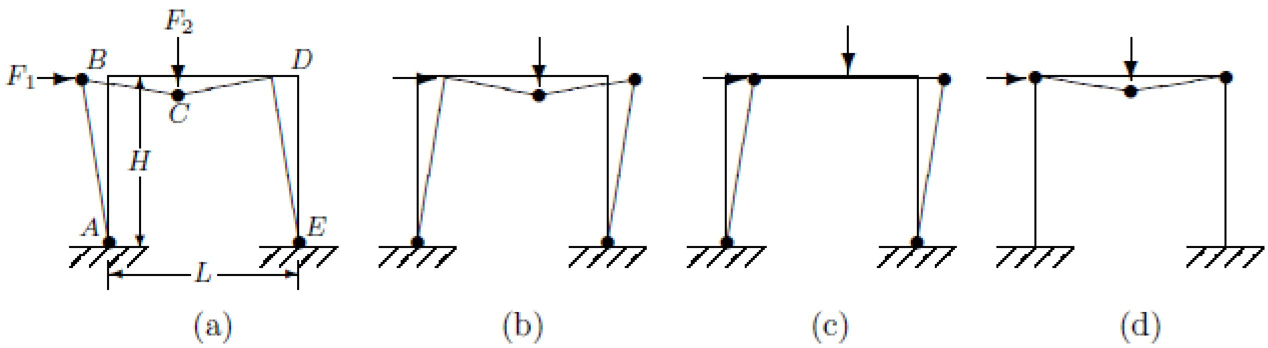


Fig. 3.35 One-story, one-bay frame

However, in figure (d), the beam mechanism is illustrated. This mechanism does not entail collapse in the sense of unlimited displacements; the deflection of the beam is limited by that of the columns and in practice the structure may be said to collapse when its displacements can become significantly greater than those in the elastic range. The only pertinent collapse mechanisms for the frame are the beam mechanism (d), the panel or sideway mechanism (c), and the composite mechanism (b), which is a superposition of (c) and (d) in which the hinge *B* is eliminated. The composite mechanism (a) – the mirror image of (b) – in which joint *D* is rigid, entails negative work done by the horizontal force and therefore is viable only when this force is zero, in which case it is equivalent to (b).

Example

Find the limit load of the frame below.

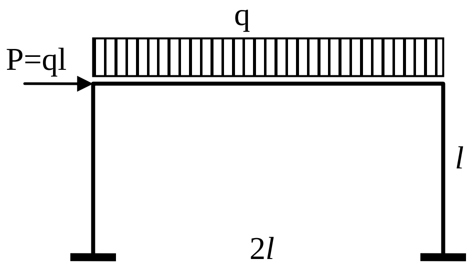


Fig. 3.37 One-story, one-bay frame

1. Kinematically admissible schemes of collapse

We verify 3 schemes of collapse: beam type, frame type and mixed:

Fig. 3.38 Mechanisms of collapse

– beam scheme

$$2\int_0^l \theta x q \mathrm{d}x = 4\overline{\overline{M}}\theta \quad \rightarrow \quad \overline{\overline{q}} = 4\frac{\overline{\overline{M}}}{l^2}$$

– frame (sideway or panel) scheme

$$ql\theta l = 4\overline{\overline{M}}\theta \quad \rightarrow \quad \overline{\overline{q}} = 4\frac{\overline{\overline{M}}}{l^2}$$

– mixed (combined, composite) scheme

$$ql\theta l + 2\int_o^l \theta x q \mathrm{d}x = \overline{\overline{M}}\theta + 2\overline{\overline{M}}\theta + 2\overline{\overline{M}}\theta + \overline{\overline{M}}\theta \quad \rightarrow \quad \overline{\overline{q}} = 3\frac{\overline{\overline{M}}}{l^2}$$

We get upper bound estimation for the smallest value from the mixed scheme:

$$\overline{\overline{q}} = \leq 3\frac{\overline{\overline{M}}}{l^2}$$

2. We check is the mixed scheme statically admissible?

We calculate:

$$H_A = -\frac{\overline{\overline{M}}}{l}$$

$$V_A = \frac{ql}{2} + \frac{\overline{\overline{M}}}{l}$$

the shear force at spandrel beam from the left:

$$Q_R = V_A - ql = \frac{\overline{\overline{M}}}{l} - \frac{ql}{2} = -\frac{\overline{\overline{M}}}{2l}$$

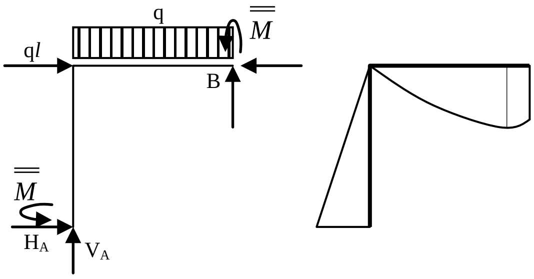


Fig. 3.39 Calculation scheme

The shear force changes the sign, the extreme value of the bending moment exceeds admissible limit value. The scheme is not admissible.
We look for the hinged section at the spandrel beam.

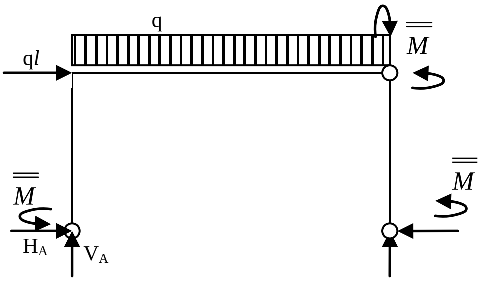


Fig. 3.40 Calculation scheme

we calculate:

$$V_A = \frac{\overline{\overline{M}}}{l} + \frac{ql}{2}, \quad H_A = \frac{2\overline{\overline{M}}}{l} - ql$$

and the shear force in the spandrel beam is:

$$Q(x) = \frac{\overline{\overline{M}}}{l} + \frac{ql}{2} - qx = 0$$

so:

$$x = \frac{\overline{\overline{M}}}{ql} + \frac{l}{2}$$

and:

$$M(x) = V_A - H_A l - \overline{\overline{M}} - \tfrac{1}{2} q x^2$$

and in the same time

$$M(x) = \overline{\overline{M}}$$

so, after the transformations, we have:

$$\frac{9}{4} q^2 - 7q \frac{\overline{\overline{M}}}{l^2} + \left(\frac{\overline{\overline{M}}}{l^2} \right)^2 = 0$$

and finally:

$$\overset{=}{q} = 2.96\frac{\overline{\overline{M}}}{l^2}.$$

3. We verify the solution by kinematic approach, assuming the kinematic scheme of collapse with the hinge at the spandrel beam is located at *a*, to the left from the middle:

$$q\theta l^2 + \int_0^{l-a}\theta q x \mathrm{d}x + \int_0^{l+a}\theta\frac{l-a}{l+a}q x \mathrm{d}x = 4\overline{\overline{M}}\theta + 2\overline{\overline{M}}\theta\frac{l-a}{l+a}$$

after simple transformations, we have:

$$q = \frac{2\overline{\overline{M}}}{l}\frac{3l+a}{(2l-a)(l+a)}$$

We calculate the extreme:

$$\frac{\partial q}{\partial a} = 0 \quad \rightarrow \quad (2l-a)(l+a) - (3l+a)(l-2a) = 0$$

we get the equation:

$$a^2 + 6al - l^2 = 0$$

with the core:

$$a = 0.162l$$

and finally:

$$\overset{=}{q} = \frac{2\overline{\overline{M}}}{l^2}\frac{3+0.162}{(2-0.162)(1+0.162)} = 2.96\frac{\overline{\overline{M}}}{l^2}.$$

The result is the same as from the static approach.

Complex frames

In a frame comprising several stories and bays, the number of possible collapse mechanisms can become quite large. Every transversely loaded member may form a beam mechanism, and each story may produce a panel mechanism. Furthermore, at any joint at which three or more members come together, a plastic hinge may form independently in each member near the joint. If only two members meet, the hinge can form in the weaker member.

It is convenient to establish a basis of independent mechanisms, called *elementary mechanisms*, such that all mechanisms may be regarded as superposition of the elementary ones. These elementary mechanisms consist of all the beam and panel mechanisms, and in addition, of the *joint mechanisms* constituted by the formation of plastic hinges, resulting in a rotation of the joint. The joint mechanisms are not in themselves collapse mechanisms, since the external work rate associated with them is zero (unless the external point moment acts at the joint), but they are used in combination with beam and/or panel mechanisms in order to cancel superfluous hinges.

Let *r* denote the degree of redundancy of the frame. A simple method of determining *r* is to cut the frame at a sufficient number of sections so that it just becomes statically determinate, that is, equivalent to a set of simply supported beams and/or cantilevers; *r* is then the number of stress resultants (moments, axial forces and shear forces) that can arbitrarily be specified at the cuts. Equivalently, *r* is the number of sections at which the moment can be arbitrarily prescribed. Suppose, now, that the number of *critical sections* – that is, sections at which a plastic hinge can form – is *n*. It follows that there are *n-r* independent relations among the *n* moments at the critical sections, and these relations are equilibrium equations. Each such equation can be associated, by means of the principle of virtual work, with a mechanism. Consequently, there are *n-r* independent mechanisms.

In the method of superposition of mechanisms, the analysis begins by determining the upper bounds predicted by the elementary beam and panel mechanisms. In order to improve the upper bound the composite mechanisms are considered. In more complex cases, it may be quite difficult to make sure that all the possible collapse mechanisms have been explored. The only way to check whether the best upper bound that has been found indeed gives the collapse load is to see if it is also a lower bound, that is, to find a statically admissible moment distribution such that bending moment is equal to cross-sectional plastic limit moment at all sections corresponding to hinges in the mechanism, and it is not greater elsewhere. A method of analysis on the lower-bound theorem is the method of inequalities and can be transformed to the problem of linear programming.

If any span carries a distributed load, then the critical section in that span must be assumed. Improvements to the upper bound can be achieved by changing the hinge locations. On the other hand, if any distributed load is replaced by a statically equivalent (equipollent) set of concentrated loads, then the collapse load calculated on the basis of the concentrated loads is a lower bound on the collapse load for the distributed load. This result is known as the *load-replacement theorem*.

Frames with inclined members, such as the gable frame, can be studied analogously.