Elastic-plastic torsion

Basic equations

We assume only shearing components of the stress tensor are not zero. The Navier's equations of internal equilibrium require:

$$\frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0,$$

and a yield criterion for perfect plasticity:

$$\tau_{xy}^2 + \tau_{xz}^2 = \sigma_0^2,$$

while the yielding limit for chosen yield criterion, is:

$$\tau_0^{HMH} = \sigma_0 / \sqrt{3}, \quad \tau_0^{CTG} = \sigma_0 / 2.$$

From the boundary conditions for Navier's equations, the resultant shear stress vector is tangent to the boundary (the product of external normal to the boundary and the stress vector is zero):

$$\tau_{xy} dz - \tau_{xz} dy = 0.$$

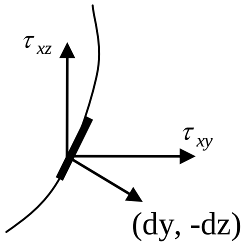


Fig. 5.1 Static boundary conditions

The equilibrium equations and the yielding criterion constitute the set of two equations with two unknowns. The problem is internally statically determined. From this follows that the stress distribution does not depend on strain range (infinitesimal or finite).

The condition for the plastic front in term of resultant shear stress is:

$$|\tau| = \sigma_0.$$

Moreover, on the front line the components of shear stress are continuous.

The above condition leads to one equation that can be established in two ways.

1. The first way uses Prandtl's stress function, similarly as in analysis in elastic range:

$$\tau_{xy} = \frac{\partial \Phi}{\partial z}, \quad \tau_{xz} = -\frac{\partial \Phi}{\partial y}.$$

The internal equilibrium equations are identities based on Schwarz theorem (unimportant order of partial derivatives for continuous functions at the point).

The yield criterion has the form:

$$\left(\frac{\partial \Phi}{\partial z}\right)^2 + \left(\frac{\partial \Phi}{\partial y}\right)^2 = \sigma_0^2$$

with boundary condition

$$\frac{\partial \Phi}{\partial y} dy + \frac{\partial \Phi}{\partial z} dz = 0,$$

which, after integration, takes the form:

$$\Phi = \text{const} = \mathrm{C}$$

For simply connected cross-section one admits $C = 0$ as a rule; however, for multiply connected regions the constant for each contour can be determined from additional conditions (for instance, in elastic range the curvilinear integral over the hole boundary from Prandtl's function derivative with respect to the normal to the contour is equal to doubled area of the hole).

2. We introduce a new variable β, which is the angle between resultant stress vector and axis *y* of the coordinate set:

$$\tau_{xy} = \sigma_0 \cos\beta, \quad \tau_{xz} = \sigma_0 \sin\beta\,,$$

and in that case the yield criterion is identically fulfilled.
The Navier's equations are the form:

$$\cos\beta \frac{\partial\beta}{\partial z} - \sin\beta \frac{\partial\beta}{\partial y} = 0\,,$$

and the boundary condition:

$$\cos\beta dz - \sin\beta dy = 0\,.$$

Let's introduce an angle α, describing the direction perpendicular to the resultant stress vector:

$$\cos\beta = \sin\alpha, \quad \sin\beta = -\cos\alpha\,,$$

so:

$$\tau_{xy} = \sigma_0 \sin\alpha, \quad \tau_{xz} = -\sigma_0 \cos\alpha\,,$$

and the Navier's equations are:

$$\frac{\partial\alpha}{\partial y}\cos\alpha + \frac{\partial\alpha}{\partial z}\sin\alpha = \frac{d\alpha}{dn} = 0\,.$$

The derivative of the angle α in direction perpendicular to the stress is equal zero. From the equation follows that along the normal line the stress is perpendicular to the normal.
Similarly as for elastic torsion, we assume the deformations compose from cross-section rotation and its warping. The displacements distribution is linear:

$$u = u(x, y, z), \quad v = -\vartheta xz, \quad w = \vartheta xy\,,$$

where ϑ is unit torsion angle. The displacement which determines the cross-section warping depends on coordinates only: $u = u(y, z)$. The strain components are:

$$\varepsilon_x = \varepsilon_y = \varepsilon_z = \varepsilon_{yz} = 0, \quad \varepsilon_{xy} = \tfrac{1}{2}\left(\frac{\partial u}{\partial y} - \vartheta z\right), \quad \varepsilon_{xz} = \tfrac{1}{2}\left(\frac{\partial u}{\partial z} + \vartheta y\right).$$

There are six equations, four of them are identically fulfilled, two remaining determine the warp function and function φ or λ. The functions can be removed for theories of total strain (HI) or flow theory for perfect plastic material (LM).

Nádai's sand hill analogy
From condition of perfect plasticity:

$$\tau_{xy}^2 + \tau_{xz}^2 = \tau_0^2$$

and Prandtl's function, we have:

$$\left(\frac{\partial\Phi}{\partial y}\right)^2 + \left(\frac{\partial\Phi}{\partial z}\right)^2 = \tau_0^2 \quad \rightarrow \quad \left|\mathbf{grad}\left(\frac{\Phi}{\tau_0}\right)\right| = 1\,.$$

The last equation means that in the plastic domain the slope of Prandtl's function surface is constant. There is a formal identity between the differential equation and boundary conditions for a stress function for torsion of a perfectly plastic prismatic bar, and those for the height of the surface of a granular material, such as dry sand, which has a constant angle of rest. This is so-called sand hill analogy.
Similarly to the elastic solution, the volume of solid limited by Prandtl's function is proportional to the torsion moment. The torsion moment one can calculate from geometric considerations only.
The lines of surfaces crossing are the lines of discontinuity. For the simple shapes of cross-section the discontinuity lines can be easily established.

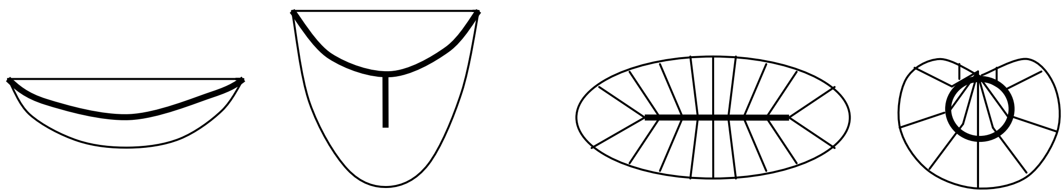


Fig. 5.2 Edges of sand hills for different shapes of cross-section

At singular points:
- convex – the lines reach the points along the bisection perpendicular
- concave – the line has the conical form, for instance for cardioid (the circle rolls on circle) the line has the circle shape.

Nádai's roof analogy
For simple connected domains, the Prandtl's membrane analogy is used in elastic region and the sand hill analogy in plastic region. For both regions the function of stress differs only that in elastic zone fulfill Poisson's equation and in plastic zone

$$\left(\frac{\partial \Phi}{\partial z}\right)^2 + \left(\frac{\partial \Phi}{\partial y}\right)^2 = \sigma_0^2$$
.

The analogy consists of rigid roof (sand hill analogy). Under it the membrane under pressure acts as the material in elastic range. Displacements of the membrane are constrained by the roof. At the beginning, the membrane does not touch the roof – the torsion is elastic. With the load increase, the membrane begins to stick to the roof in the plastic zones. For full plastic load, the membrane sticks to the roof everywhere.

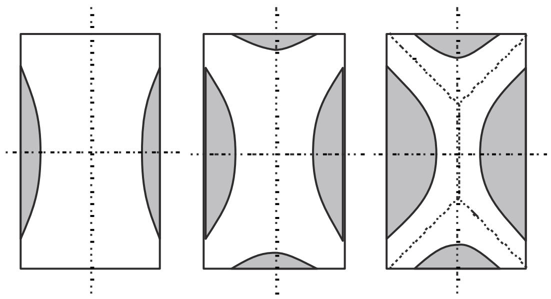


Fig. 5.3 Nádai's roof analogy

Sadovsky's analogy
This is some generalization of the roof analogy onto multiply connected domains. We build the cylinders over the holes with their shapes and we pour sand on the cross-section. Next, we lower the cylinders to the moment when the sand pours into the cylinder at the whole perimeter. This fulfills the condition of constant axial displacement on the contour of the hole.

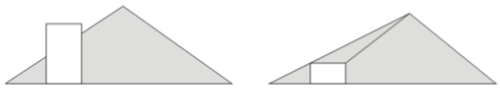
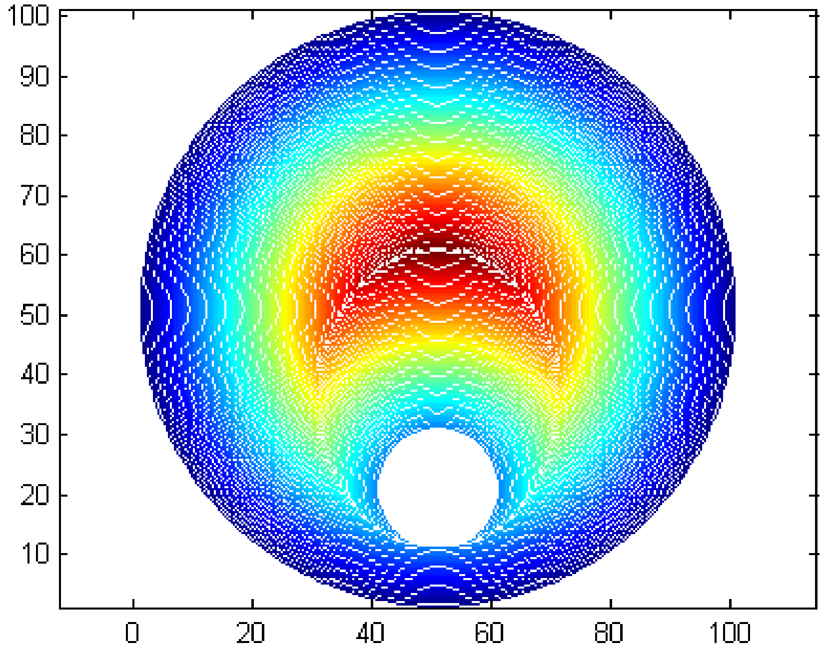


Fig. 5.4 Sadovsky's analogy for multi connected domain



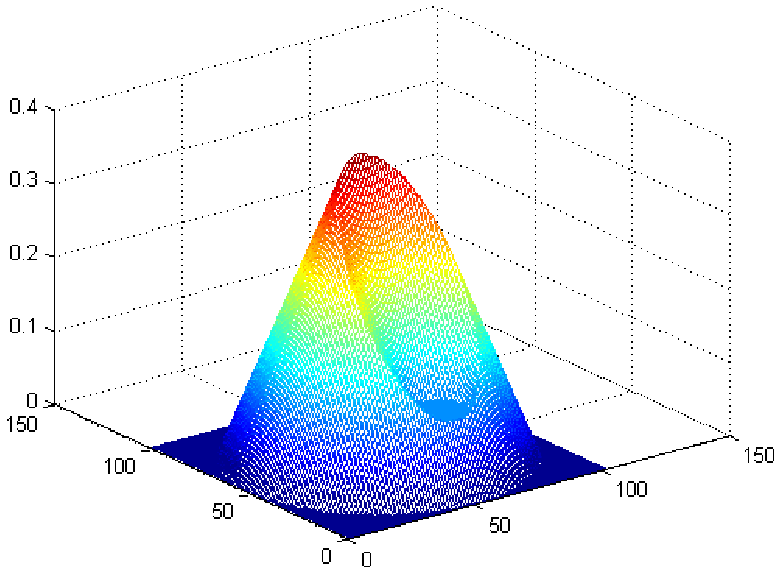


Fig. 5.5 Sadovsky's analogy – cd: contour lines

Cross-section warping

We seek the form of Prandtl's function. In elastic zone it is:

$$\frac{\gamma_{xy}}{\gamma_{xz}} = \frac{\tau_{xy}}{\tau_{xz}}.$$

In plastic zone, from Hooke's equations we can write in rates:

$$d\gamma_{xy} = \frac{1}{G} d\tau_{xy} + d\varphi \tau_{xy},$$

$$d\gamma_{xz} = \frac{1}{G} d\tau_{xz} + d\varphi \tau_{xz}.$$

Because of constant stress in plastic zone, their rates are zero and:

$$d\gamma_{xy} = \frac{\tau_{xy}}{\tau_{xz}} d\gamma_{xz},$$

so, the relation is identical as in elastic case:

$$\frac{\gamma_{xy}}{\gamma_{xz}} = \frac{\tau_{xy}}{\tau_{xz}}.$$

Substituting the strain components, we get the differential equation for axial displacement:

$$\tau_{xz}\left(\frac{\partial u}{\partial y} - \vartheta z\right) - \tau_{xy}\left(\frac{\partial u}{\partial z} + \vartheta y\right) = 0,$$

where the shear stress components are known functions. The solution of the partial differential equation is quite easy.

Assuming for the yielding:

$$\tau_{xy} = \sigma_0 \sin\alpha, \quad \tau_{xz} = -\sigma_0 \cos\alpha$$

we have:

$$\left(\frac{\partial u}{\partial y} - \vartheta z\right)\cos\alpha + \left(\frac{\partial u}{\partial z} + \vartheta y\right)\sin\alpha = 0$$

and:

$$\frac{du}{dn} = \vartheta(-y\sin\alpha + z\cos\alpha).$$

The term in brackets is the distance from the gravity center to the normal of the point s. In plastic zone:

$$\frac{du}{dn} = \vartheta s.$$

If – as it is usually assumed – the warping at the gravity center is zero, this is the same at every line that pass through the gravity center and not by discontinuity lines. The warping is also equal zero at every axis of the cross-section.

Circular cross-section

In this particular case, the solution is simpler due to lack of the warping. The strain radial distribution is linear.

$$\gamma_{x\theta} = \vartheta r$$

and stress distribution is linear in elastic zone:

$$\tau_{x\theta} = G\vartheta r$$

and constant in plastic zone:

$$\tau_{x\theta} = \tau_0.$$

The radius of the plastic front we calculate from the compatibility condition at the zones border:

$$r_{pl} = \tau_0 / G\vartheta.$$

The figure below presents the stress radial distribution.

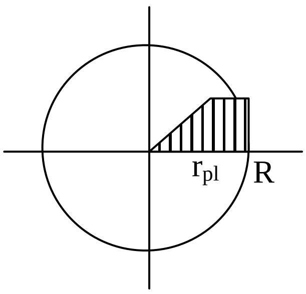


Fig. 5.6 Stress distribution for circular cross-section

The torsion moment is given by the integral:

$$M_x = \iint_A \tau_{x\theta} dA = 2\pi\int_0^R \tau_{x\theta} r^2 dr = 2\pi\left(G\vartheta\frac{r_{pl}^4}{4} + \tau_0\frac{R^3 - r_{pl}^3}{3}\right) = \frac{\pi}{6}\tau_0\left(4R^3 - \frac{\tau_0^2}{G^3\vartheta^3}\right).$$

The unit torsion angle is function of torsion moment:

$$\vartheta = \frac{\tau_0}{GR\sqrt[3]{4}} \sqrt[3]{\frac{1}{1-\frac{3M_x}{2\pi\tau_0 R^3}}}.$$

and, if the moment tends to the plastic limit value:

$$\overline{\overline{M}}_x = \tfrac{2}{3}\pi R^3 \tau_0,$$

the unit angle tends to infinity.
The limit radius is:

$$r_{pl} = R\sqrt[3]{4\left(1-\frac{3M_x}{2\pi\tau_0 R^3}\right)}$$

Example – unloading and residual stress
Let’s consider torsion of a cross-section in a shape of ring with radii ratio:

$$\beta = {R_i}/{R_e} \quad < 1$$

Similarly as for a circle, there is no warping of the cross-section and the shearing stress distribution is given by the formulae:

$$\tau_{x\theta} = \begin{cases} G\vartheta r & \left(R_i \le r \le r_{pl}\right) \\ \tau_0 & \left(r_{pl} \le r \le R_i\right) \end{cases}$$

From the condition of stress conformity, we get:

$$M_x^{el} = \iint_A \tau_{x\theta} r \mathrm{d}A = \frac{2}{3}\pi\tau_0 R_e^3 \left[1-\frac{1}{4}\left(\frac{r_{pl}}{R_e}\right)^3 - \frac{3}{4}\frac{R_i}{r_{pl}}\beta^3\right]$$

and corresponding limit values are:

$$\overline{M}_x = M_x\left(r_{pl} = R_e\right) = \tfrac{1}{2}\pi\tau_0 R_e^3\left(1-\beta^4\right), \quad \overline{\vartheta} = \frac{\tau_0}{GR_e},$$

$$\overline{\overline{M}}_x = M_x\left(r_{pl} = R_i\right) = \tfrac{2}{3}\pi\tau_0 R_e^3\left(1-\beta^3\right), \quad \overline{\overline{\vartheta}} = \frac{\tau_0}{GR_i}.$$

For the solid bar $\left(R_i \to 0\right)$ we have:

$$M_x^{el} = \frac{2}{3}\pi\tau_0 R^3\left[1-\frac{1}{4}\left(\frac{r_{pl}}{R}\right)^3\right]$$

$$\overline{M}_x = M_x\left(r_{pl} = R\right) = \tfrac{1}{2}\pi\tau_0 R^3, \quad \overline{\vartheta} = \frac{\tau_0}{GR},$$

$$\overline{\overline{M}}_x = M_x\left(r_{pl} = 0\right) = \tfrac{2}{3}\pi\tau_0 R^3, \quad \overline{\overline{\vartheta}} \to \infty.$$

From the elastic relations for unloading process:

$$\tilde{e}_{ij} - e_{ij} = \frac{1}{2G}\left(\tilde{s}_{ij} - s_{ij}\right)$$

we determine the residual stress in totally unloaded cross-section:

$$\tau_{x\theta}^0 = \begin{cases} G\vartheta^0 r & \left(R_i \le r \le r_{pl}\right) \\ \tau_0 - Gr\left(\vartheta - \vartheta^0\right) & \left(r_{pl} \le r \le R_e\right) \end{cases}$$

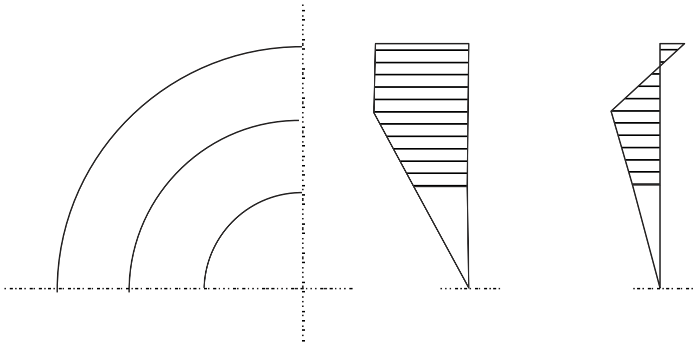


Fig. 5.7 Stress distribution in twisted ring

and the residual unit torsion angle can be calculated provided that the torsion moment is zero:

$$M_x^0 = 0 \quad \rightarrow \quad \vartheta^0 = \vartheta \left\{ 1 - \frac{4 r_{pl}}{3 R_e \left(1-\beta^4\right)} \left[1 - \frac{1}{4} \left(\frac{r_{pl}}{R_e} \right)^3 - \frac{3}{4} \frac{R_i}{r_{pl}} \beta^3 \right] \right\}.$$

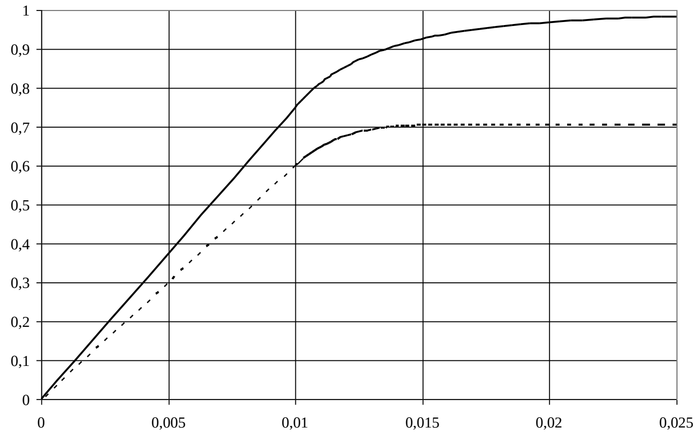


Fig. 5.8 Torsion moment as function of unit twist angle (solid line – circle, dashed line – ring)

Calculation example: a ring

$R_e = 0{,}03$ m, $R_i = 0{,}025$ m, $\tau_0 = 150$ MPa, $G = 80$ Gpa

elastic limit moment: $\overline{M} = 3{,}293$ kNm

plastic limit moment: $\overline{\overline{M}} = 3{,}574$ kNm

loading moments: $\overline{M} < M_1 = 3{,}4$ [kNm], $M_2 = 3{,}5$ [kNm] $< \overline{\overline{M}}$

position of yielding front: $r_{pl\_1} = 0{,}02892$ [m], $r_{pl\_2} = 0{,}027536$ [m]

unit twist angle: $\vartheta_1 = 0{,}0648$ [1/m], $\vartheta_2 = 0{,}0681$ [1/m]

after unloading: $\vartheta_1^0 = 0{,}000317$ [1/m], $\vartheta_2^0 = 0{,}001679$ [1/m]

stress: $\tau_1(R_i) = 0{,}634, \quad \tau_1(r_{pl}) = 0{,}733, \quad \tau_1(R_e) = -4{,}84$ [MPa]

$\tau_2(R_i) = 3{,}36, \quad \tau_2(r_{pl}) = 3{,}70, \quad \tau_2(R_e) = -9{,}39$ [MPa]

For thicker ring: $R_i = 0{,}01$ [m]:

$\overline{M} = 6{,}283$ kNm, $\overline{\overline{M}} = 8{,}168$ kNm, we assume: $M = 7{,}736$ kNm, for which: $r_{pl} = 0{,}02$ [m]

$\vartheta = 0{,}0938, \quad \vartheta^0 = 0{,}0168$

shear stress: $\tau(R_i) = 13{,}4, \quad \tau(r_{pl}) = 27{,}9, \quad \tau(R_e) = -34{,}7$ [MPa]

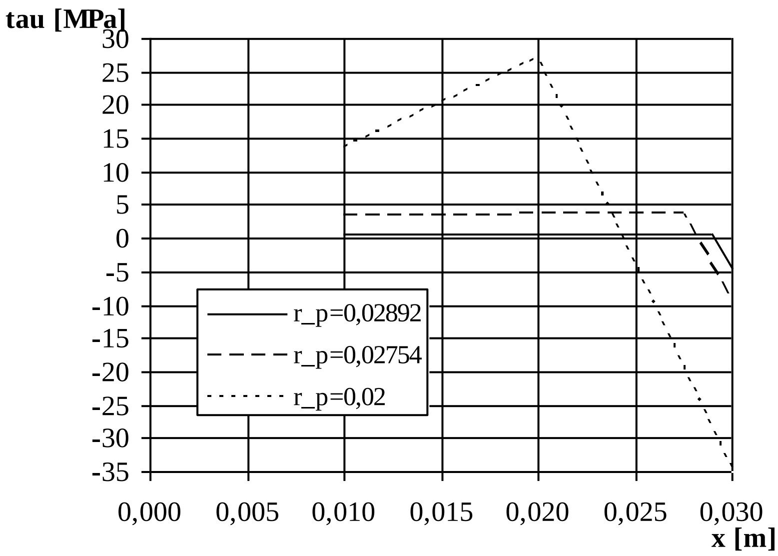


Fig. 5.9 Residual stress in ring cross-section

Limit analysis of plates

Basic definitions and equations

The generalized cross-section forces in rectangular plate, calculated for unit length of the plate midline, are (in order of increasing importance):

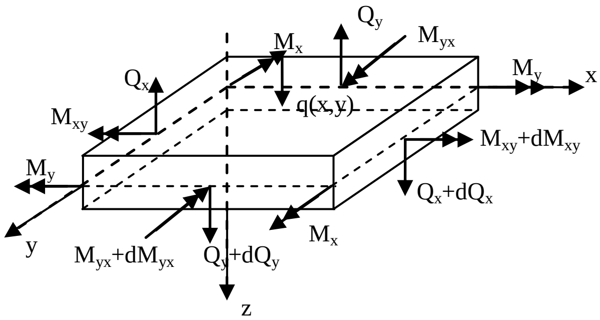


Fig. 5.10 Plate – cross-sectional forces

Cross-section forces in rectangular plate

– membrane forces (usually neglected, do not sum upon *i* index)

$$N_i = \int_{-h}^{h} \sigma_{(i)(i)} dz$$

– shear forces (their influence on yielding usually neglected)

$$Q_i = \int_{-h}^{h} \sigma_{iz} dz$$

– torsion moments

$$M_{ij} = \int_{-h}^{h} \sigma_{ij} z dz$$

– bending moments (the main factor of the problem)

$$M_i = \int_{-h}^{h} \sigma_{(i)(i)} z dz$$

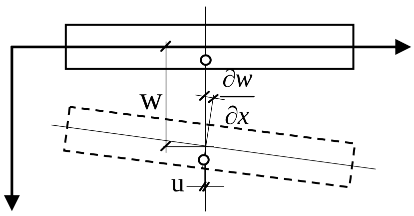


Fig. 5.11 Love's-Kirchhoff's hypothesis

Assuming the Love-Kirchhoff hypothesis (the segments normal to the middle surface remain normal and straight), cf. fig., and after additional simplifications:

- the influence of the shear forces on yielding is neglected
- the membrane forces and torsion moments are neglected (we consider only the bending),
- infinitesimal strain,

The displacement and strain states written in term of deflection function are:

$$\begin{cases} \dot{\varepsilon}_x = -z\dfrac{\partial^2 \dot{w}}{\partial x^2} = \dot{\kappa}_x z \\ \dot{\varepsilon}_y = -z\dfrac{\partial^2 \dot{w}}{\partial y^2} = \dot{\kappa}_y z \\ \dot{\varepsilon}_{xy} = -2z\dfrac{\partial^2 \dot{w}}{\partial x \partial y} = \dot{\kappa}_{xy} z \end{cases},$$

where x, y, z are dimensionless coordinates (relative to the plate height), $\dot{w}$ – dimensionless velocity of the deflection, and symbols $\dot{\kappa}$ are dimensionless rates of curvature changes.
The other components of the strain velocity are zero.
The exact solution in general case is complicated. We limit ourselves to the kinematically admissible strain fields, it means to the upper bound limit.

Theory of yield lines

We divide a plate into rigid regions connected by lines of angular velocity discontinuity, which are also referred as the plastic hinge lines or yield lines.
We assume that:

- plastic flow concentrates along lines where the bending moments have extreme values (limit plastic value of the bending moment),
- the deflection derivative in direction normal to the yield lines jumps (plastic hinge), while in the parallel direction is continuous
- the regions between yield lines are plane and rigid,
- the collapse can be also in the form of continuous field of yield lines, that makes straight-line fan.

On the yield line, we have:

$$\dot{\kappa}_n = -\frac{\partial^2 \dot{w}}{\partial n^2} \to \infty, \quad \dot{\kappa}_t = -\frac{\partial^2 \dot{w}}{\partial t^2} \quad \text{(limited)} \ , \quad \dot{\kappa}_{nt} = 0,$$

and the yield lines agree with trajectories of the rate of principal curvatures, $\dot{\kappa}_{nt} = 0$.
To obtain the dependence between cross-section forces and the rates of generalized strain, we use the Levy-Mises theory:

$$\dot{\varepsilon}_x = \tfrac{2}{3}\dot{\lambda}\left(\sigma_x - \tfrac{1}{2}\sigma_y\right),$$

$$\dot{\varepsilon}_y = \tfrac{2}{3}\dot{\lambda}\left(\sigma_y - \tfrac{1}{2}\sigma_x\right),$$

$$\dot{\gamma}_{xy} = 2\dot{\lambda}\tau_{xy}.$$

In limit state for perfectly plastic material, we get:

$$M_x = \sigma_x h^2, \quad M_y = \sigma_y h^2, \quad M_{xy} = \tau_{xy} h^2.$$

The yield criterion HMH is:

$$m_x^2 - m_x m_y + m_y^2 + 3m_{xy}^2 = 1,$$

with dimensionless moments (per unit length) relative to the limit moment in uniaxial stress state:

$$M_0 = \sigma_0 h^2.$$

Using the yield criterion, after simple considerations, we get finally:

$$m_n = 2m_t.$$

In the principal directions, we have:

$$M_n = \frac{2\sigma_0 h^2}{\sqrt{3}}, \quad M_t = \frac{\sigma_0 h^2}{\sqrt{3}}.$$

Application of the virtual work principle

We have:

$$\boxed{\mu^k \left[\iint_A q\dot{w}dA + \sum_{i=1}^{r} P_i \dot{w}_i \right] = M_n \sum_{j=1}^{h} \dot{\theta}_j l_j},$$

where μ^k is multiplier not less than exact value, q is load density, P_i point forces (i = 1...r), l_j the length of the yield line (j = 1...h).

Let's consider two cases of the yield lines.

Concentrated yield lines

This is the case of polygonal plate.

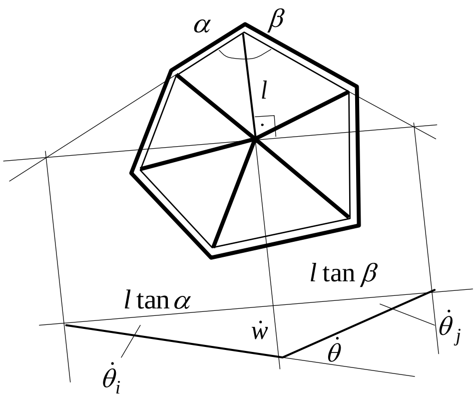


Fig. 5.12 Concentrated yield lines

From the figure, we have:

$$\dot{\theta}_i = \frac{\dot{w}}{l} ctg\alpha, \quad \dot{\theta}_j = \frac{\dot{w}}{l} ctg\beta, \quad \dot{\theta} = \dot{\theta}_i + \dot{\theta}_j.$$

From the principle of virtual work, we get:

$$P = M_n \sum_{i=1}^{n} \left(\mathrm{ctg}\,\alpha_\mathrm{i} + \mathrm{ctg}\,\beta_\mathrm{i} \right).$$

For regular n side polygon with the point force at the center, we get:

$$P^{(n)} = 2nM_n \tan\frac{\pi}{n}.$$

For $n \to \infty$ we get the limit for circular plate:

$$P^{(\infty)} = 2\pi M_n,$$

where:

$$M_n^{HMH} = 2M_0/\sqrt{3}, \quad M_n^{TG} = M_0.$$

and the fractured surface adopts a cone form.

Continuous yield lines

The fan of straight yield lines appears for the plate with curvilinear border. It can appear for polygonal plate with the number of sides less than 5 near to the corners.

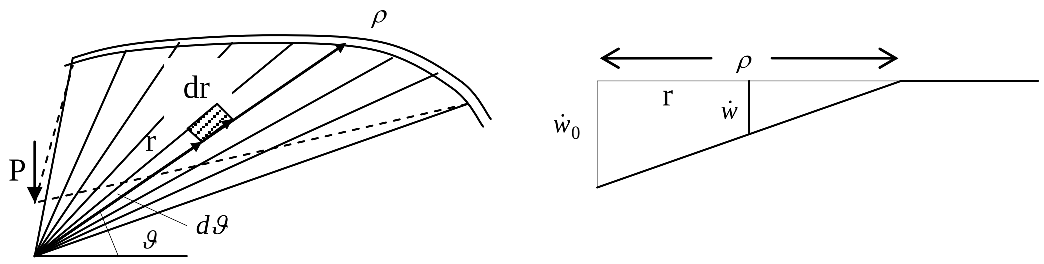


Fig. 5.13 Fan of yield lines

The upper bound limit capacity of the plate we calculate from the principle of virtual work, but the calculus is rather cumbersome. If a plate with a curvilinear contour $p = p(\theta)$, is supported along a line $r = r(\theta)$, and is loaded at the origin of polar coordinate set, the plastic bearing capacity of the plate is given by the formula:

$$P = \bar{\bar{M}} \int_0^{2\pi} \frac{p}{r}\left(1 + 2\frac{(r')^2}{r^2} - \frac{r''}{r}\right) d\theta$$

where the primes signify the derivatives with respect to variable θ.
In the plates fixed along their contour, besides the radial lines of bend that radiates from the point of the force application, there is a perimetric bend line. Denoting the radial limit moment by $\bar{\bar{M}}_r$ and the circumferential limit moment by $\bar{\bar{M}}_\theta$, the formula for the limit force is:

$$P = \bar{\bar{M}}_r \int_0^{2\pi} \left(1 + 2\frac{(r')^2}{r^2} - \frac{r''}{r}\right) d\theta + \bar{\bar{M}}_\theta \int_0^{2\pi} \left(1 + \frac{(r')^2}{r^2}\right) d\theta$$

which in the case of isotropy reduces to:

$$P = \bar{\bar{M}} \int_0^{2\pi} \left(2 + 3\frac{(r')^2}{r^2} - \frac{r''}{r}\right) d\theta$$

Circumferential bend line may arise either at the fixed end of plate or inside the area, so some parts of plate remain not destructed. The equation of a perimetric bend line can be deduced from the condition of minimum of functional:

$$J = \int_0^{2\pi} F(r, r', r'') d\theta = \int_0^{2\pi} \left(2 + 3\frac{r'^2}{r^2} - \frac{r''}{r}\right) d\theta = \min$$

A curve $r = r(\theta)$, for which the integral attains minimum, should fulfill the Euler's equation:

$$\frac{\partial F}{\partial r} - \frac{d}{d\theta}\left(\frac{\partial F}{\partial r'}\right) + \frac{d^2}{d\theta^2}\left(\frac{\partial F}{\partial r''}\right) = 0$$

After differentiating, we get a differential equation:

$$r'^2 - rr'' = 0$$

which a solution is two-parameters family of logarithmic spirals with an origin at the point of the force application:

$$r = Ae^{C\theta}$$

where *C* and *A* are constants. In a special case there are concentric circles. The circumferential yield line should be closed for plates fixed over whole boundary, so we get the collapse scheme in the form of a circle inscribed inside the boundary, and limit bearing capacity independently to the plate shape is:

$$P = 2\bar{\bar{M}} \int_0^{2\pi} d\theta = 4\pi\bar{\bar{M}} = 12.57\bar{\bar{M}}$$

This value cannot be surpassed by any value for another support.

Sometimes, when a circumferential bend line crosses simply supported boundary of a plate, a symmetric forms of bend lines arise. It can be proved that minimum value of the bearing capacity corresponds to the angle of 45 degrees between the extreme radii of yielding fan surface and the boundary.

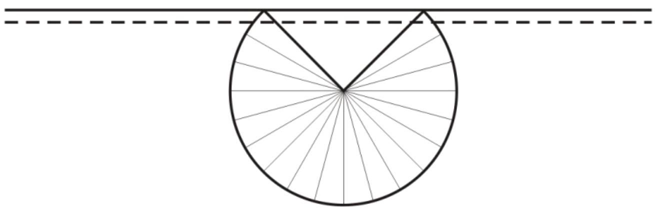


Fig. 5.14 Yielding fan nearby the simply supported boundary

Example
The circular plate is simply supported on some part of the border.

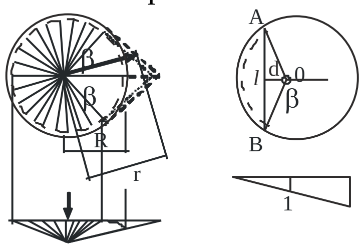


Fig. 5.15 Circular plate partially supported.

R stands for the radius of the plate, r the distance of fictitious support to the plate center. For conical part of yield lines, we have (constant contour radius):

$$\overline{\overline{P}} = \overline{\overline{M}} \int_{\beta}^{2\pi-\beta} d\vartheta$$

and for the rib:

$$\overline{\overline{P}} = \overline{\overline{M}} \frac{R}{r_{\max}} 2\mathrm{ctg}\,\alpha = \overline{\overline{M}} \frac{R}{r_{\max}} 2\tan\beta .$$

Total capacity is sum of both terms because the force should trigger both mechanisms off. Viewing that:

$$r_{\max} = \frac{R}{\cos\beta}$$

we have

$$\overline{\overline{P}} = 2\overline{\overline{M}}\left(\pi - \beta + \sin\beta\right).$$

The collapse has the conical form for the supported part of the plate and the form of two planes with the rib between.
For $\beta = 0$ we get solution as for the plate supported on the whole perimeter:

$$\overline{\overline{P}} = 2\pi\overline{\overline{M}} .$$

For $\beta = \pi/2$ we have:

$$\overline{\overline{P}} = 2\overline{\overline{M}}(\pi/2 + 1) = 5.146\overline{\overline{M}}$$

If $\beta > \pi/2$, the collapse form may be different, for example along the yield line AB, fig. 6b. From the figure, it follows:

$$\frac{l}{2d} = \tan(\pi - \beta) = -\tan\beta .$$

The limit value for this scheme is:

$$\overline{\overline{P}} = -2\overline{\overline{M}} \tan\beta \, .$$

The limit case we obtain comparing two solutions:

$$2\overline{\overline{M}}(\pi - \beta + \sin\beta) = -2\overline{\overline{M}} \tan\beta \quad \to \quad \beta = 2{,}03 \quad (=116°) \, .$$

Example

Determine the bearing capacity of triangular plate loaded by a point force at the center of equilateral triangle, see figure.

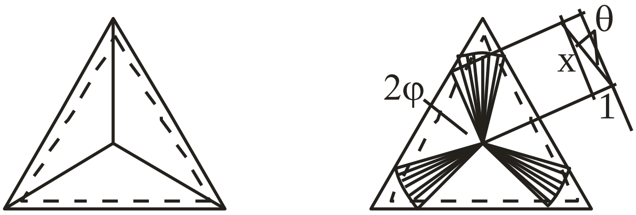


Fig. 5.16 Yielding schemes for the triangular plate

Yielding schemes of triangular plate

(a) The case of three straight lines

For the scheme from the left we get from the general formula:

$$P = 6\sqrt{3}\overline{\overline{M}} = 10.39\overline{\overline{M}}$$

(b) The case of three conical sectors

From the isotropic plate formula augmented by the value corresponding to the formation of six radial yield lines, we have:

$$P(\varphi) = \left[\int_0^{2(\pi-3\varphi)} 2d\theta + 6l\theta \right] \overline{\overline{M}}$$

where θ is the rotation angle of the triangular parts connected to the plate center:

$$\theta = \frac{\cot\varphi}{l}$$

Substituting the last formula to the integral, we get:

$$P(\varphi) = [4(\pi - 3\varphi) + 6\cot\varphi]\overline{\overline{M}}$$

Minimum value will be attaint for:

$$\frac{\partial P}{\partial \varphi} = 0 \to \overline{\overline{P}} = (\pi + 6)\overline{\overline{M}} = 9.14\overline{\overline{M}}$$

The last value is less than the first one, so, the final answer is the last value.