Plane problems[1]

Introduction

The concepts of plastic flow and plastic collapse were regarded as essentially equivalent, representing a state in which a body continues to deform under constant applied forces. In practical applications, however, the two concepts have quite different meanings.
Plastic collapse describes undesirably large deformations of an already formed body that result from excessive forces.
The concept of plastic flow, on the other hand, is usually applied to the deliberate forming of a mass of solid (such as metal or clay) into a desired shape through the application of appropriate forces.
These two large classes of problems, of fundamental importance in mechanical and civil engineering, can be attacked by the same methodology – the theory of rigid-perfectly plastic bodies, with the help of the theorems of limit analysis.

Plane strain state

This is such material flow, that:

– vectors of strain velocities are parallel to a plane and independent on one coordinate
– vectors are not depending on the distance from the plane.

We can find such state in uniform prismatic body, with the length much greater than the cross-section dimensions, when the constraints imposed onto displacements are identical for each cross-section and loads act perpendicularly to the cross-section and are the same for each cross-section.
It is worth to add, that the problem here is different from the plane strain state problem in theory of elasticity. Mathematical methods adopted here are rather similar to these applied to plane flow of liquid.
We adopt the scheme of the rigid-plastic body. There is difficult to estimate the error of the assumption, nevertheless without the simplification the solution becomes much more complicated. The simplified solution is characterized by rigid regions between elastic ones.

Basic equations

From the displacements' functions:

$$u_x = u_x(x,y), \quad u_y = u_y(x,y), \quad u_z = 0 ,$$

follows that $\varepsilon_z = 0$. From the H-I theory as well as from the L-M theory (similarity between strain deviators with its velocities and the stress deviators) with the incompressibility condition, we get:

$$\sigma_z - \sigma = 0, \quad \sigma = \tfrac{1}{2}(\sigma_x + \sigma_y) .$$

It is known that σ_z is a principal stress. Other principal stress values can be determined from the formulas:

$$\left.\begin{matrix}\sigma_{\max} \\ \sigma_{\min}\end{matrix}\right\} = \frac{\sigma_x + \sigma_y}{2} \pm \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2} = \sigma_z \pm \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2} .$$

It is obvious, that σ_z is intermediate principal value. In that case, the shear stress will be:

$$\tau_{\max} = \frac{1}{2}(\sigma_{\max} - \sigma_{\min}) = \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2} \equiv \tau ,$$

and, using the notation, we can write down:

$$\sigma_1 = \sigma + \tau, \quad \sigma_z = \sigma, \quad \sigma_2 = \sigma - \tau ,$$

which means, that the stress state at every point is sum of hydrostatic pressure σ and pure shear stress, τ.
Knowing the principal directions we know, that the maximum shear stress is attained on the planes inclined by 45° to the principal directions.

Slip lines

A slip line is a line tangent to the surface of extreme shear stress. It is obvious that the two families of orthogonal lines exist:

[1] adapted from Lubliner, op. cit.

$x = x(\alpha, \beta), \quad y = y(\alpha, \beta)$,

where α, β are parameters, and on the line α the parameter β is constant and vice versa. An infinitesimal element, localized between slip lines is equally tensioned in the slip lines directions.

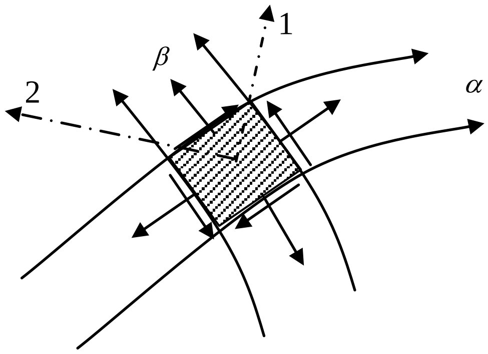

Fig. 6.1 Slip lines and principal directions

Plastic state is attained for:

$\tau = const = \tau_s, \quad \sigma_{\max} - \sigma_{\min} = 2\tau_s$.

When we take into consideration the equilibrium equations with the yielding criterion and static boundary conditions are imposed on the boundary, we get a statically determined problem, which solution doesn't depend on the strains. The solution of statically undetermined problems is much more complicated. For this reason so-called semi-inverse method are commonly used. There are searched such slip lines that calculated field of the strain rates fulfill the kinematic boundary conditions. Although the approximate solution is limited, many important engineering problems have been solved in this way.

From the equation of the stress state

$$\left.\begin{aligned} \sigma_x &= \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2}\cos 2\alpha_{1x}, \\ \sigma_y &= \frac{\sigma_1 + \sigma_2}{2} - \frac{\sigma_1 - \sigma_2}{2}\cos 2\alpha_{1x}, \\ \tau_{xy} &= \frac{\sigma_1 - \sigma_2}{2}\sin 2\alpha_{1x}, \end{aligned}\right\}$$

and general form of the yielding criterion (HMH as well as CTG):

$$(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2 = 4\tau_0^2$$

and substituting for the fractions sums of σ and differences of τ_0, we have for the angle $\theta = \alpha_{1x} - \frac{\pi}{4}$:

$$\left.\begin{aligned} \sigma_x &= \sigma - \tau_0 \sin 2\theta, \\ \sigma_y &= \sigma + \tau_0 \sin 2\theta, \\ \tau_{xy} &= \tau_0 \cos 2\theta, \end{aligned}\right\}$$

for which the yielding criterion is fulfilled.

When we put obtained formulas into the equilibrium equations, we get the set of differential equations with partial derivatives and hyperbolic type.

Characteristics

Let's consider a set of two equations with partial derivatives first order of two searched functions *U, V*:

$$A_1U_{,x} + B_1U_{,y} + C_1V_{,x} + D_1V_{,y} = E_1,$$

$$A_2U_{,x} + B_2U_{,y} + C_2V_{,x} + D_2V_{,y} = E_2,$$

where the coefficients $A_1 \ldots E_2$ are functions of two variables x, y.

If the coefficients depend on variables x, y only, the set is called linear. When the right hand sides of the equations are zero, the set is homogeneous. For linear and homogeneous equation set the superposition principle (the additivity theorem) is valid. If the coefficients depend also on the functions U, V, the set is called quasi-linear, to emphasize that the superposition principle is no longer valid but the dependence on the functions U,V is linear.

Let's suppose, that the solution of the equations set is known along a line *L*, given in parametric form $y=y(s), x=x(s)$. A fundamental question arises: can the values of the functions U,V be found nearby the line *L*? The answer is positive, if values of four derivatives can be determined:

$U_{,x}, U_{,y}, V_{,x}, V_{,y}$.

The original set of equations with the derivatives of any two combinations of solutions

$U_{,x}\dot{x}+U_{,y}\dot{y}=\dot{U},\quad V_{,x}\dot{x}+V_{,y}\dot{y}=\dot{V},$

constitutes a set of 4 equations which solution depends on its determinant.

If the determinant is not equal to zero, the derivatives are uniquely determined by the line *L* equation and values of the functions U,V along the line.

If the determinant is equal to zero there are infinitely many solutions to the equation set and the derivatives are not determined uniquely. The curves with such properties are called the characteristics of the set.

Writing down the condition of determinant zeroing we arrive to the equation:

$$\Delta \equiv a\left(\frac{\dot{y}}{\dot{x}}\right)^2 - 2b\frac{\dot{y}}{\dot{x}} + c = 0$$

The quantity

$$m=\frac{\dot{y}}{\dot{x}}=\frac{dy/ds}{dx/ds}=\frac{dy}{dx}$$

determines the slope of the curve *L*. Solving the second order equation, we get:

$$m_1=\frac{1}{a}\left(b+\sqrt{b^2-ac}\right),\quad m_2=\frac{1}{a}\left(b-\sqrt{b^2-ac}\right).$$

If the *L*-line slope attains one of the values at a point, there is no unique solution to the problem.

So, there are three possibilities:

1. the equation is of the elliptic type and there is no real roots,
2. in the case $b^2-ac=0$ there is one characteristic direction and the set is parabolic
3. two roots of equation determine two characteristic directions and the set is of hyperbolic type.
4. In the hyperbolic region the lines tangent to the characteristic directions make curvilinear net, called the characteristics' net. The characteristics equations take the form:
5. $dy=m_1dx$ (first characteristics family)
6. $dy=m_2dx$ (second characteristics family)
7. In the case of the linear set of equations, the characteristics' net is fixed. In the opposite case, the characteristics' net is different for each particular problem.
8. If a function *v* is known at one point of a characteristic curve, then *dv* can be calculated for a neighboring point on this curve, and continuing this operation allows *v* to be determined at all points on the curve. If *v* is known at all points of a curve ∑ that is nowhere tangent to the characteristic, then its values may be calculated along all the characteristics that intersect ∑.
9. If two characteristics emanating from different points of ∑ should intersect at some point *Q* of the x_1x_2 plane then, in general two different values of *v* will be obtained there. The point *Q* is therefore the locus of a discontinuity in *v*.
10. Consider, now, a curve ∑ that is nowhere tangent to a characteristic, and suppose that a discontinuity (jump) in $v_{,1}$ or $v_{,2}$ occurs at a point *P* of ∑. Since the only information necessary to determine the characteristic curve through *P* and the values of *v* along this curve is the value of *v* at *P*, the directional derivative of *v* along the characteristic will have the same value on either side of it. Consequently, if any jump in $v_{,2}$ or $v_{,1}$ is propagated through the x_1x_2 plane, it must occur across the characteristic through *P*.
11. The characteristics are the lines, along which we can glue analytic solutions. Crossing the characteristics the derivatives of solution may have discontinuities. The equations of characteristics have simple physical explanation: they are the slip lines. The net of slip lines is the most relevant element of the solution to the plane strain state; thus, the plane problem reduces in some sense to the problem of construction correct net of the slip lines.

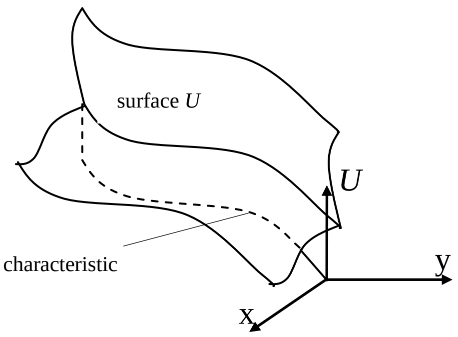

Fig. 6.2 Weak discontinuity of solution along a characteristic

Characteristics properties

For the hyperbolic system we have:

$$\frac{dy}{dx} = m_1 = \tan\left(\varphi + \frac{\pi}{4}\right) \text{ (along a line } \alpha)$$

$$\frac{dy}{dx} = m_2 = \tan\left(\varphi - \frac{\pi}{4}\right) \text{ (along a line } \beta)$$

The equations can be converted into equivalent form of Hencky's:

$$\sigma + 2\tau_0\varphi = C_\alpha \text{ (along } \alpha)$$

$$\sigma - 2\tau_0\varphi = C_\beta \text{ (along } \beta),$$

which are simply the equilibrium equations along the characteristics.

The characteristics have some important properties, discovered mainly by Hencky.

1. Along the slip lines the pressure changes proportionally to the angle between the slip line and *x* axis. This follows directly from Hencky's equations.
2. First theorem of Hencky. An angle between lines tangent to the two slip lines belonging to one family, and emanating from the points of intersection with a slip line of the other family, not depend on the choice of the last one and is constant.

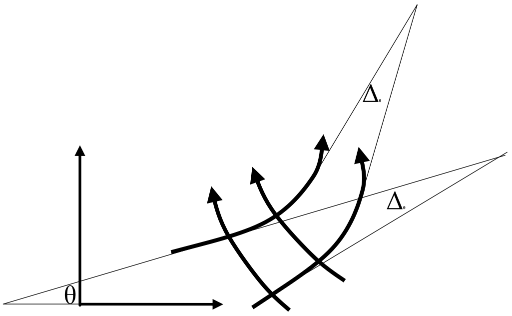

Fig. 6.3 First theorem of Hencky

3. If pressure value σ is known at any point of the slip net, it can be calculated at an arbitrary point of the field.

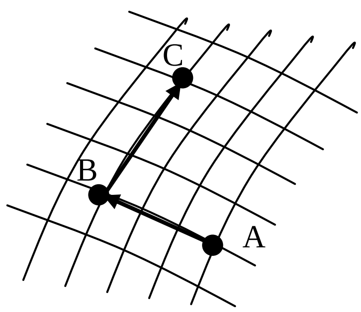

Fig. 6.4 Pressure σ calculation at a field point

4. If a characteristic is straight line, values of pressure and angle (and of stress state components in consequence) are constant along the line. Thus, in regions with straight slip lines the stress state is homogeneous.
5. If a segment of slip line belonging to one family is a straight line, all respective segments of the family cut by lines of second family will be straight also.

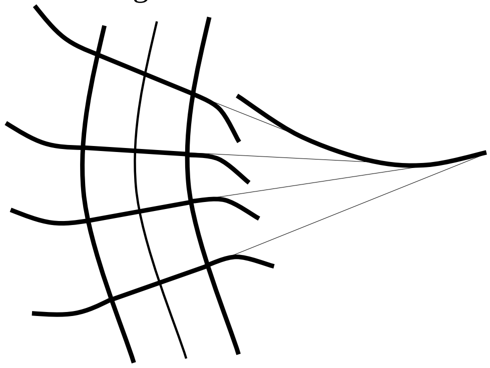

Fig. 6.5 Slip lines similarity

6. The straight segments cut by the slip lines of second family of slip lines have the same length. The curvature centers of the slip lines of the first family lie on the second family's involute.
7. Second theorem of Hencky's. Moving along a slip line the curvature radius of the second slip lines family changes by the length of covered arc (like unreeling a rope from the involute).
8. When moving along the slip line in direction of greater curvature, the curvature radius diminishes.
9. (Prandtl's theorem). If the envelope degenerates into one point, all slip lines crossing the pole form a fan of slip lines with constant curvature radius with the center at the pole.
10. If moving along one slip lines family we come across on the point of stress derivatives' discontinuity, the line is a line of discontinuity of curvature radii of the second family of the slip lines. In that way the net of orthogonal slip lines may be composed from pieces of different analytic functions.

Many different properties of the characteristics are not exploit in the theory of plasticity.

Traction boundary conditions

Traction boundary-value problems may be of three types, with the construction of characteristics corresponding to each type shown in the figure below.

1. Cauchy problem. The boundary is nowhere tangent to a characteristic. The existence and uniqueness of the solution has been proved inside quadrilateral formed by the characteristics families.
2. Riemann's problem. The boundary is composed of characteristics of both families and conformity conditions on the boundary are fulfilled. The existence and uniqueness of the solution has been proved inside quadrilateral formed by the characteristics families.
3. Mixed problem. A part of the boundary agrees with a characteristic (with the conformity conditions) and other boundary part crosses the characteristics only once and a linear combination of solution is fulfilled. The existence and uniqueness of the solution has been proved inside the triangle.

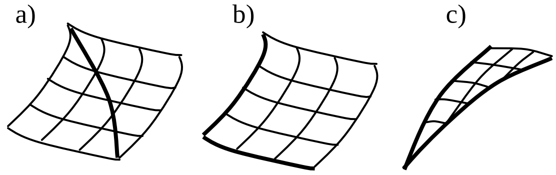

Fig. 6.6 Cauchy's, Riemann's and mixed problems

Problems of the bearing capacity

The deformation process usually proceeds in two stages; in the first stage a body behaves as elastic and after passing some level of loads the plastic zones arise in the body. The yielding is slowed down by the elastic zones. This is the stage of contained yielding. At the end of the stage the yielding strains localize and become rapidly growing when the elastic ones change insignificantly. At some value of the load localized plastic zones cause rapid raise of strains. It is third stage of uncontained plastic flow.

Assuming a rigid-plastic model of a body, it can be possible to obtain results very convergent to the experimental data. In particular it is necessary:
1. determine the border between the stiff zone and plastic one, that is a slip line or its envelope,
2. determine a stress field which fulfill the equations of equilibrium, yield criterion and boundary conditions,
3. prove, that the rigid zone can remain rigid; it means that stress field extension onto stiff zone should be found to fulfill the equilibrium equations, yield criterion and boundary conditions,
4. find solution for the velocities consistent with kinematic boundary conditions and conditions onto the boundary between stiff and plastic zones,
5. check non-negativity of yielding function at any point of the plastic zones.

If all conditions are fulfilled the solution is called complete. The value of the bearing capacity found is a real value.
The solution which fulfills the conditions 1- 3 is called a static solution. The value found is not greater than a real one.
The condition 3 is difficult to meet. The solution which don’t fulfill this condition is called a kinematic solution. The value found is not less than a real one. The solution technique starts with assumption of a field of characteristics and next we try to fulfill the conditions 1-5.

Velocity fields

If a boundary value problem is solved by constructing a characteristic network, over a part of the region representing the body, then the loading forms an upper bound to that under which plastic flow becomes possible, since, as will be shown below, a kinematically admissible velocity field can then be found. The exact flow load is found when the stress field can be extended in a statically and plastically admissible manner into the rigid region.
A plane whose coordinates are the velocities along the characteristics is known as the hodograph plane, and a diagram in this plane showing the velocity distribution is called a hodograph. It follows from the characteristic relations that the Hencky-Pradtl properties apply to the hodograph as well. A rigid region, if it does not rotate, is represented by a single point in the hodograph plane.
It must be remembered, however, that the flow rule actually gives only the ratios among the strain rates. Another interpretation of this result is that some strain rates are infinite, meaning that the tangential velocity component is discontinuous across a characteristic (the normal component must be continuous for material continuity), that is, if slip occurs. The characteristics are thus the potential loci of slip and are therefore also called slip lines. Kinematically admissible velocity fields with discontinuities across slip lines are often used in the construction of solution. In particular, slip may occur along a characteristic forming the boundary between the plastic and rigid regions.
In this manner many engineering problems have been solved, often with surprisingly good accuracy:
- squeezing a rigid stamp into half-space (three solutions: by Prandtl, Hill and Prager-Hodge)
- squeezing a stamp into strip of limited width
- cutting a strip by two stamps
- cut wedge under a stamp
- wedge (acute and obtuse) under uniform pressure
- traction of a strip with notches (rounded and sharp)
- pull broaching of a strip through a rigid die (with straight and curvilinear walls)

For example, for a stamp driven into a half-space, several solutions have been obtained with different assumptions. In Hill’s solution triangles of uniform state under a stamp don’t overlap as it is in other solutions. In Prandtl’s solution it is assumed, that the triangular region under a stamp moves in conformity with the stamp. Prager-Hodge’s solution describes an intermediate case.

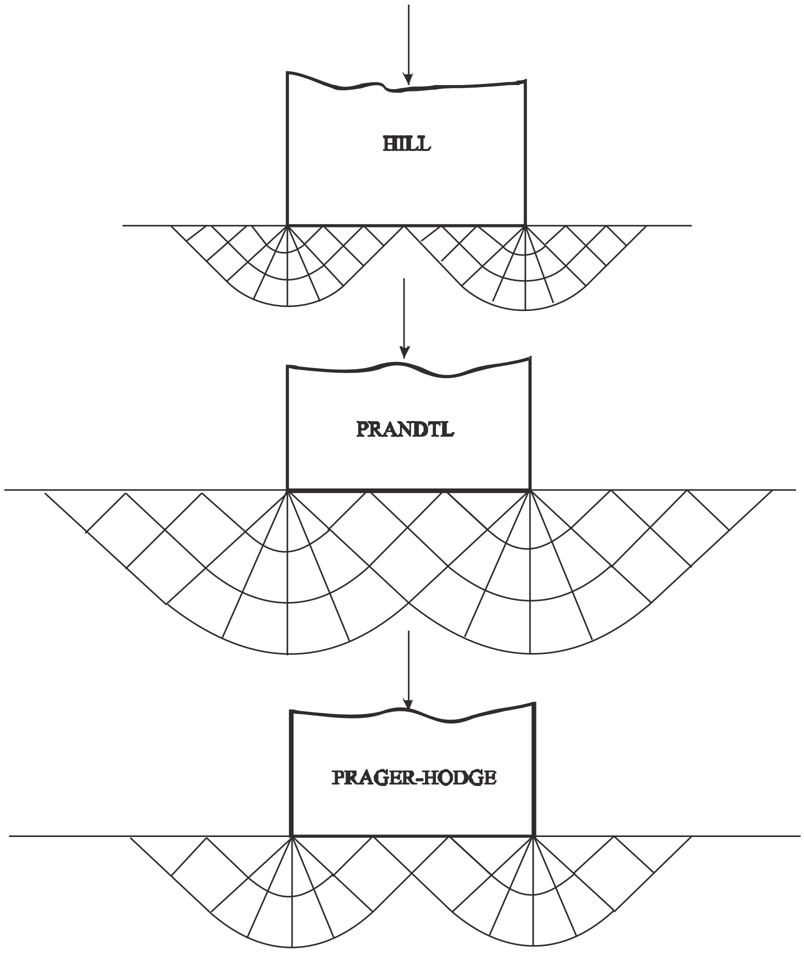

Fig. 6.7 Stamp – different solutions

In some reasons the Hill's solution appears to be the best:
- better corresponds with intuitive ideas of yield zones progress
- shape of a border between plastic and rigid zones unambiguously determines a velocity field
- gives the least extension of plastic movement.

In the triangular zone under the stamp, for the Cauchy's problem, the straight characteristics means homogeneous stress state. Intuitively we assume that it is uniaxial compression. We have:

$$\theta = -\tfrac{3}{4}\pi, \quad \sigma = -\tau_0 .$$

In the rest part, a corner of the stamp is a point of stress discontinuity and a point of start of a characteristics fan. This is the Riemann's problem. For polar fan, from conformity of the stress with the previous zone follows:

$$-\tfrac{3}{4}\pi < \theta < -\tfrac{\pi}{4}, \quad \sigma = -\tau_0\left(1 + \tfrac{3}{2}\pi + 2\theta\right) .$$

In the triangular part under the stamp there is homogeneous stress state as previously; from conformity condition of stress state with the fan part, we have:

$$\theta = -\tfrac{\pi}{4}, \quad \sigma = -\tau_0(1+\pi) .$$

In this way the boundary value problems was solved. Distribution of pressure below the stamp is presented on a figure below.

The stress components in coordinate set x, y are determined, substituting θ, σ into the formulae, as:

$$\left.\begin{aligned} \sigma_x &= \sigma - \tau_0 \sin 2\theta, \\ \sigma_y &= \sigma + \tau_0 \sin 2\theta, \\ \tau_{xy} &= \tau_0 \cos 2\theta, \end{aligned}\right\}$$

we get:

$$\sigma_{xx} = -\tau_0 \pi, \quad \sigma_{yy} = -\tau_0(2+\pi), \quad \tau_{xy} = 0 .$$

The stress σ_{yy} below the stamp is equal to the stamp pressure. It follows that the stress distribution is uniform. Therefore the bearing capacity is:

$$\overline{\overline{P}} = 2a\tau_0(2+\pi),$$

where $2a$ is the stamp width.

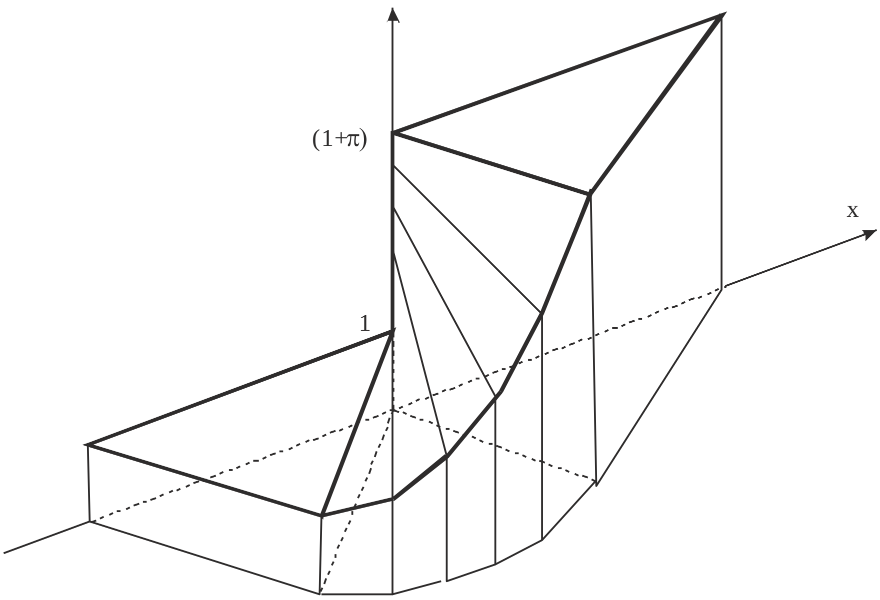

Fig. 6.8 Pressure under a stamp (Hill's solution)

Shakedown

Cycle in the theory of plasticity

A cycle is a process after which the variables return to the original values. In plastic materials, due to irreversibility of phenomena, there is no cycle, except some particular situations.
For this reason, the theory of plasticity uses the concept of a quasi-cycle. As a quasi-cycle we understand a deformation process at the termination of which only the independent variables (exertion factors) return to their initial values. The dependent variables may behave during each quasi-cycle in a different manner, and so it is difficult to speak of repeatable effects.
Two typical quasi-cycles can be distinguished.. If the stresses can be treated as the exertion factors, then we shall speak about a stress-quasi-cycle. Similarly a strain-quasi-cycle may be introduced, see figure below.

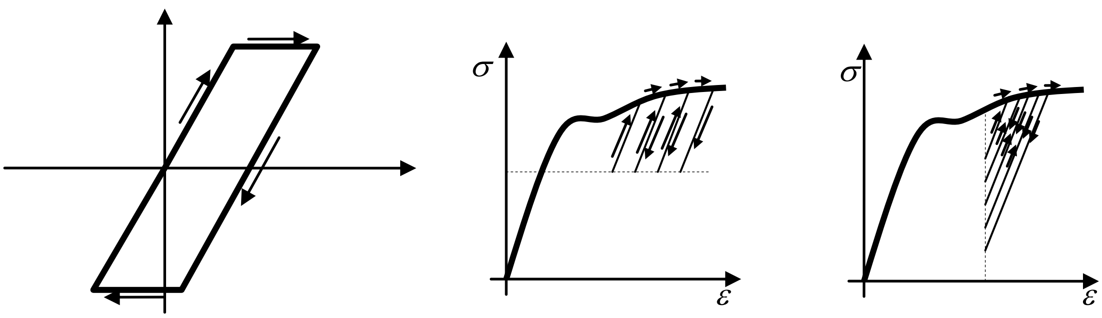

Fig. 6.9 Cycle, stress-quasi-cycle and strain-quasi-cycle

Shakedown theorem

An elastic-plastic structure subjected to a cyclic loading program may fail due to excessive deformation even though the extreme values of the load may be lower than that corresponding to plastic collapse.
The plastic strains produced in a critical load cycle progressively increase as the cycle is repeated, and the structure becomes unserviceable. In some cases, cycles of plastic flow may occur in alternating directions causing an early fatigue failure. The structure will be safe, however, if the load varies in such a way that the

stress distribution remains within the yield limit after an initial period of plastic flow. The structure is then said to shake down to a state of residual stress, the change in strain being entirely elastic as the load is subsequently varied between the prescribed limits.

Melan's theorem (static shakedown criterion)

An elastic-perfectly plastic structure will shake down for given extreme values of the load if there exists a time-independent distribution of residual stress that nowhere leads to stresses beyond the yield limit when superimposed on the stress distribution corresponding to the elastic response of the structure.

$$f\left[\sigma_{ij}^{res}(x) + \Delta\sigma_{ij}^{e}(x,t)\right] \leq \sigma_H(x)$$

$$\sigma_{ij,j}^{res} = 0 \quad (\text{in } V) \qquad \sigma_{ij}^{res} n_j = 0 \quad (\text{at } S_T)$$

The theorem says nothing about the amount of plastic deformation that may occur before shakedown state is attained.

Koiter's theorem (kinematic non-shakedown criterion)

An elastic-perfectly plastic structure will not shake down for given extreme values of the load if there exists a kinematically admissible velocity field, with a rate of plastic strains and stress, that a power exerted during a cycle surpasses a power dissipated within the structure.

$$\int_0^T dt \iint_{S_T} p_j \dot{u}_j^k dS > \int_0^T dt \iiint_V \sigma_{ij}^k \dot{\varepsilon}_{ij}^{kp} \, dV$$

The Koiter's theorem is a kinematic counterpart of Melan's theorem. An extension of Melan's theorem to work-hardening materials is due to Mandel.

Example

A statically undetermined structure is loaded by two independently acting forces, see figure below.

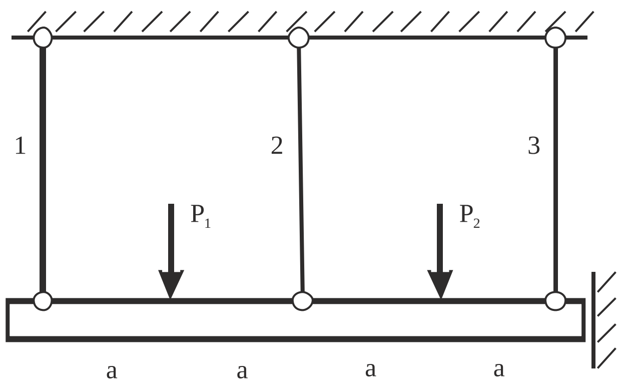

Fig. 6. 10 Simple structure under cyclic loads

We introduce dimensionless variables:

$$p_i = \frac{P_i}{\sigma_0 A},$$

from two static equations and one kinematic equation, we get:

$$n_1 = \frac{7}{12} p_1 + \frac{1}{12} p_2,$$

$$n_2 = \frac{1}{3}(p_1 + p_2),$$

$$n_3 = \frac{1}{12} p_1 + \frac{7}{12} p_2.$$

Assuming that the bars yield in turn, $n_i = 1$, we obtain the interaction curves (in the form of straight lines), that we can draw knowing the points of intersection:

(-2,2), (1.5,1.5), (-1.5,-15), (2,-2).

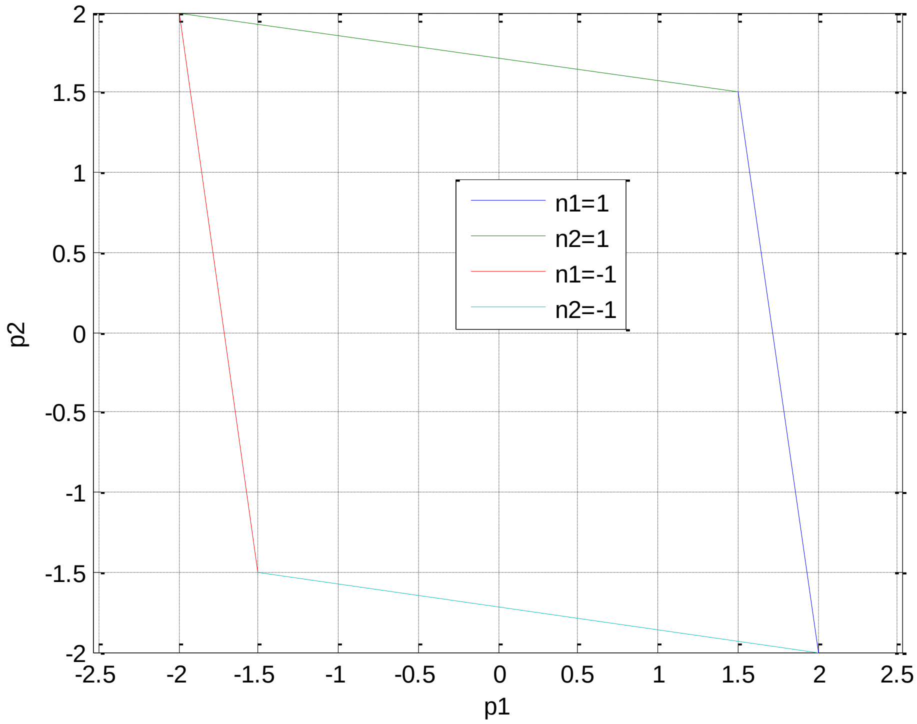

Fig. 6.11 Elastic interaction curves

We consider 6 mechanisms of plastic collapse. A calculation of plastic bearing capacity gives us other intersection points: these are the edges of plastic limit curves:
(-1.5, 2.5), (1.5, 1.5), (2.5, -1.5), (1.5, -2.5), (-1.5, -1.5), (-2.5, 1.5).
Each corner corresponds with the state where two bars or three are in the ultimate state. Only in the pure tension and pure compression three bars are in plastic state.
The analysis of plastic bearing capacity gives another set of limit lines. The corners correspond with the ultimate state of three bars each time. From top counterclockwise the corners are: TTC (tension-tension-compression), T0C (0 = zero force), TCC, CCC, CCT, C0T, CTT, TTT.

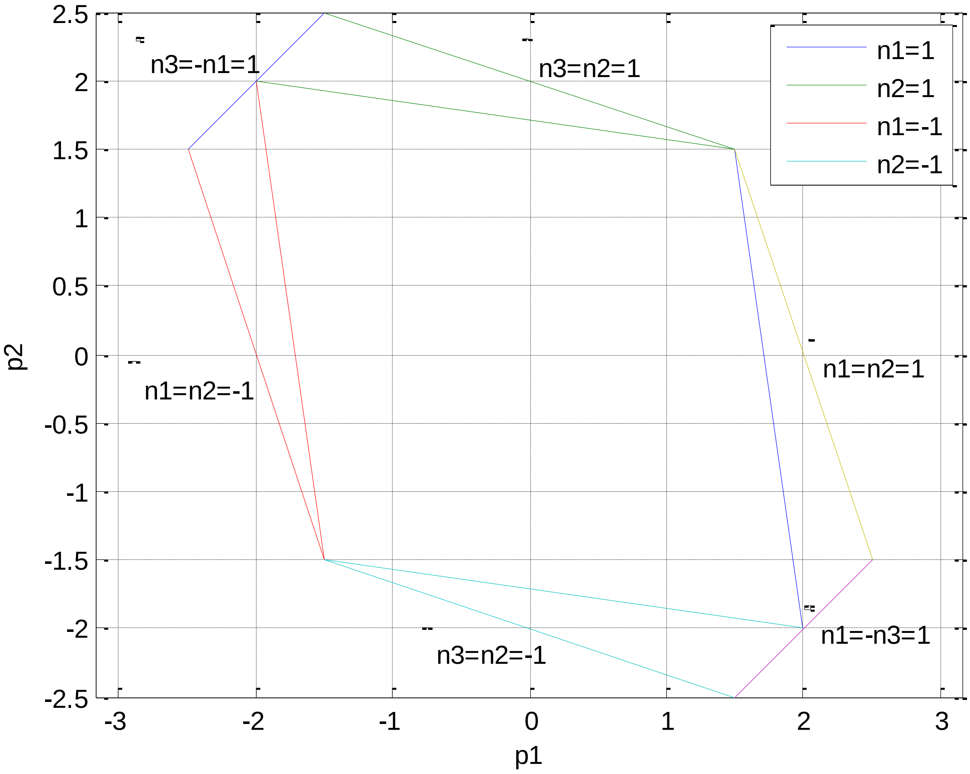

Fig. 6.12 Plastic and elastic interaction curves

Modified yielding surface

Let's consider three loading programs, where variables p_1, p_2 change in a cyclic manner, see a table below:

period		0	1	2	3	4	5	6	7	8
cycle 1	p_1	0	13/7	0	0	0	13/7	0	0	0
	p_2	0	0	0	13/7	0	0	0	13/7	0
cycle 2	p_1	0	13/7	0	0	0	13/7	0	0	0
	p_2	0	0	0	-13/7	0	0	0	-13/7	0
cycle 3	p_1	0	13/7	0	-1/7	0	13/7	0	-13/7	0
	p_2	0	0	0	0	0	0	0	0	0

In every cycle in the first stage the stress surpasses the elastic range:

$$\tilde{\sigma}_1 = \sigma_0, \quad \tilde{\sigma}_2 = \frac{11}{14}\sigma_0, \quad \tilde{\sigma}_3 = \frac{1}{14}\sigma_0$$

and after unloading the residual stresses form a self-equilibrated set of forces:

$$\rho_1 = -\frac{1}{12}\sigma_0, \quad \rho_2 = \frac{1}{6}\sigma_0, \quad \rho_3 = -\frac{1}{12}\sigma_0 .$$

Onto the residual stress field, we impose fictitious elastic stress field that corresponds to allowable changes of exertion forces. In this way, we determine a modified surface of the flow. In accordance with the Melan's theorem we get the region of the structure shakedown:

$$-\frac{11}{12} \le \frac{7}{12}p_1 + \frac{1}{12}p_2 \le \frac{13}{12},$$

$$-\frac{5}{6} \le \frac{1}{3}p_1 + \frac{1}{3}p_2 \le \frac{5}{6},$$

$$-\frac{11}{12} \le \frac{1}{12}p_1 + \frac{7}{12}p_2 \le \frac{13}{12}.$$

We determine the intersection points of the modified yield surface, see figure below.
To calculate the stress and strain states we adopt the procedure as follows.

First, we assume an elastic process. If the limit stress is exceeded, it means the plastic strain in one bar. The process, when two bars are in plastic state corresponds with the plastic limit state and uncontrolled increase of strain. The cycle may be stable if and only if one bar is in yielding state each time. Thus, in the middle bar the yield state will not be achieved. So, for the stress calculation we use the formulas, using dimensionless quantities p, s, and e:

$s_1 + s_3 = \frac{2}{3}n$, $s_1 - s_3 = 2m$, where $n = p_1 + p_2$, $m = \frac{1}{4}(p_1 - p_2)$

so, for the elastic state: $s_1 = \frac{1}{3}n + m,\ s_2 = \frac{1}{3}n,\ s_3 = \frac{1}{3}n - m$

Second, superimposing elastic stress with the residual one, we check the yield limit is not exceeded. If it is, we reduce the stress to the yield limit. For the elastic-plastic states:

$$s_1 > 1 \xrightarrow{yields} s_1 = 1, \quad s_2 = n + 2m - 2, \quad s_3 = 1 - 2m$$
$$s_1 < -1 \xrightarrow{yields} s_1 = -1, \quad s_2 = n + 2m + 2, \quad s_3 = -1 - 2m$$
$$s_3 > 1 \xrightarrow{yields} s_3 = 1, \quad s_1 = 2m + 1, \quad s_2 = n + 2m - 2,$$
$$s_3 < -1 \xrightarrow{yields} s_3 = -1, \quad s_1 = 2m - 1, \quad s_2 = n + 2m - 2$$

Third, from the elastic strain components we calculate strains, superimposing them with the residual strains, if any. For the third strain we use the compatibility condition.

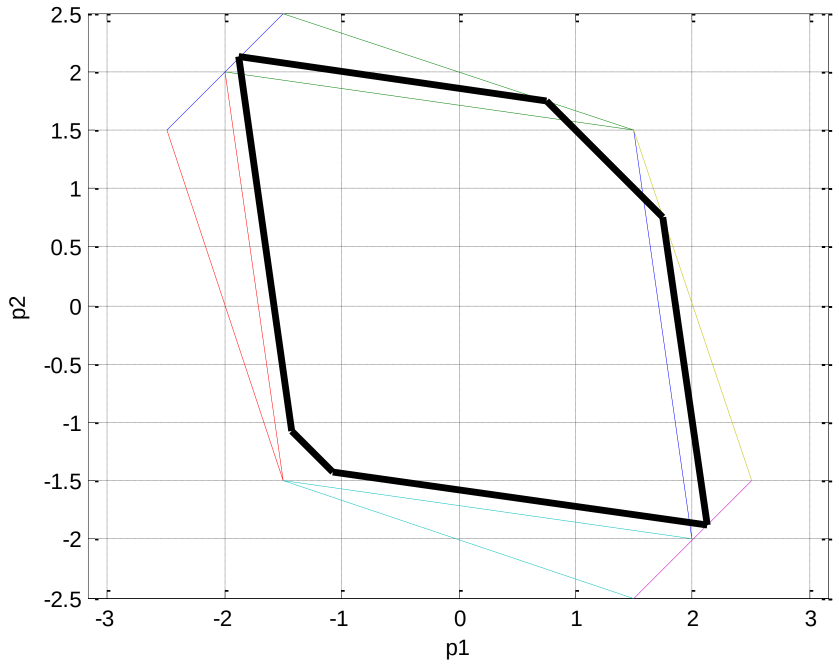

Fig. 6.13 Modified surface of flow

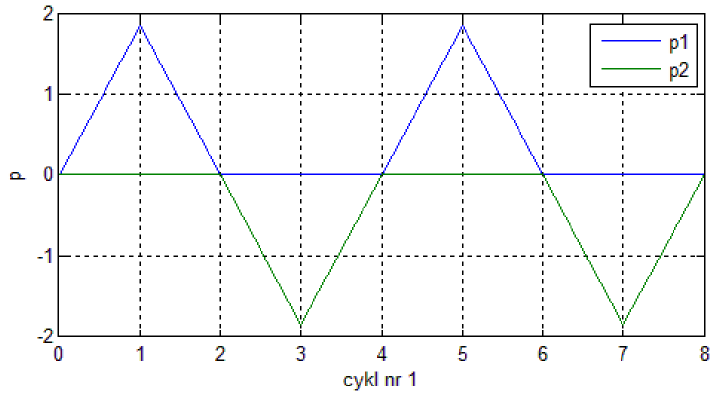

Fig. 6.14 Cycle no 1

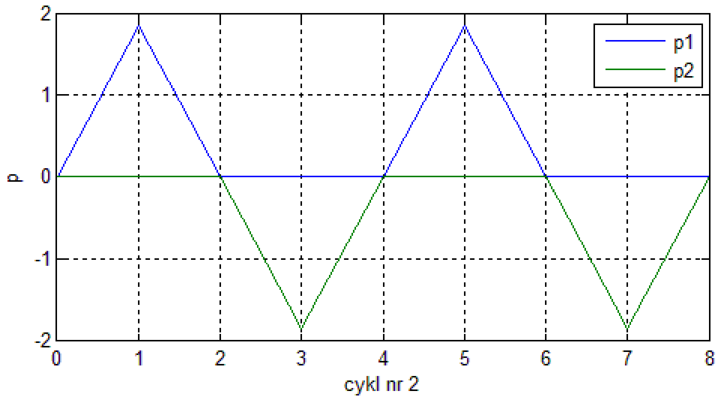

Fig. 6.15 Cycle no 2

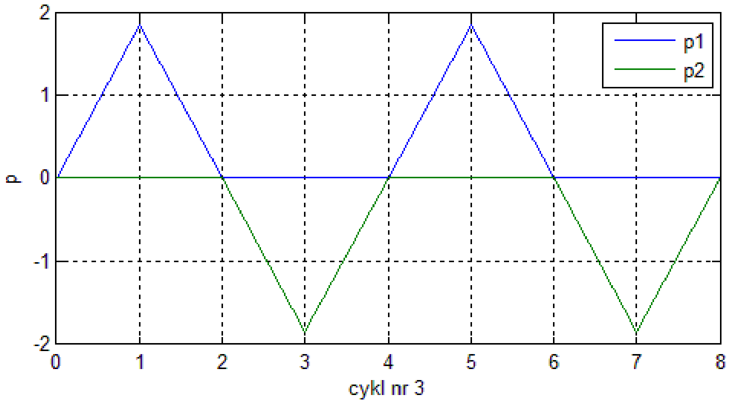

Fig. 6.16 Cycle no 3

The shakedown region contains the cycle no 1 completely. The structure response is entirely elastic: after the first period, the bars sustain the elastic strains that don't surpass double elastic strain limit. The changes of bar strains are presented in the figure below.

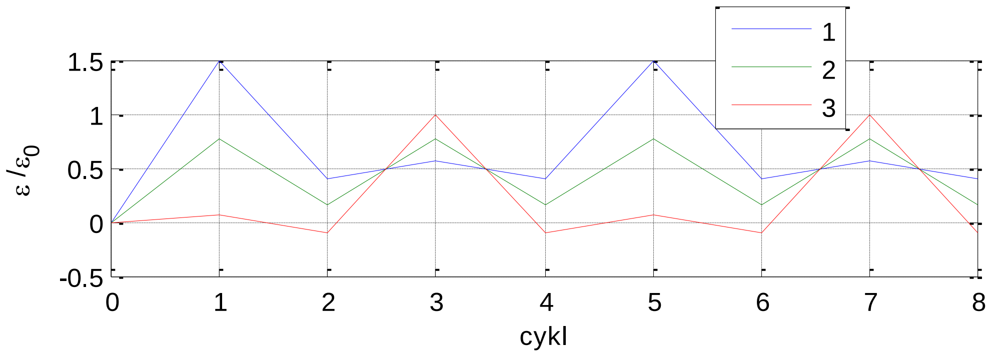

Fig. 6.17 Structure response in cycle no 1

The structure cannot adapt to the loading program no 2, and the unlimited rise of elastic strains follow the first period. The changes of bar strains are presented in the figure below.

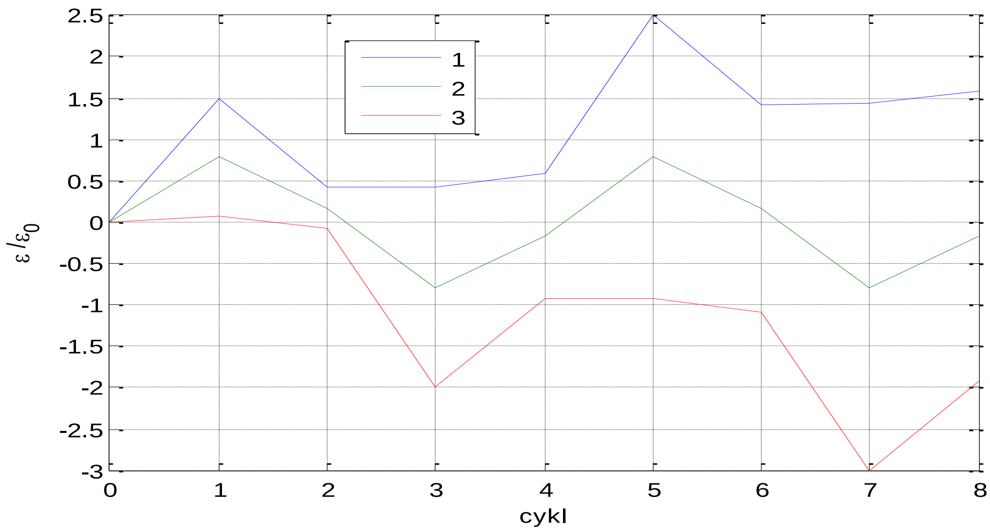

Fig. 6.18 Structure response in cycle no 2

When one of the external forces changes the sense of action and the second is zero, the structure cannot shakedown to the loading due to alternating plastic strains of tension and compression in the bar no 1. This phenomenon of alternating plastic strains leads to a low-cycle fatigue of structure. Unlimited dissipation of energy results from the alternating plastic strains that cancel each other out, so , the total plastic strain may remain not large.

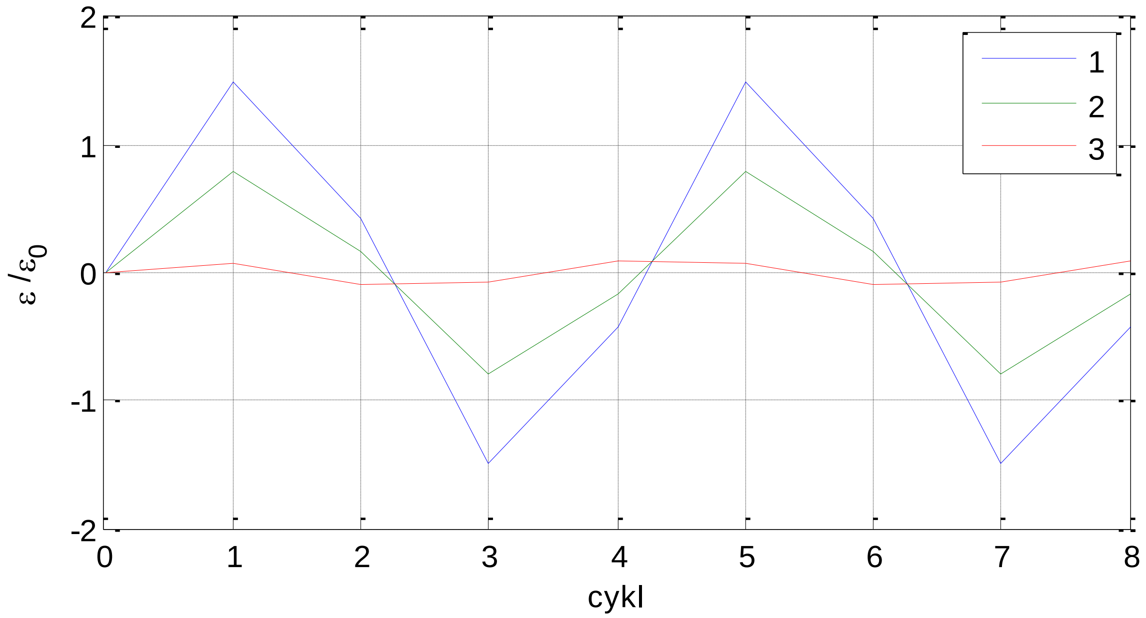

Fig. 6.19 Structure reponse in cycle no 3