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SPECIFICATION OF ENERGY-BASED CRITERION OF ELASTIC LIMIT STATES FOR CELLULAR MATERIALS

SPECYFIKACJA ENERGETYCZNEGO KRYTERIUM SPRĘŻYSTYCH STANÓW GRANICZNYCH DLA MATERIAŁÓW KOMÓRKOWYCH

The aim of the paper is to apply the energy-based criterion of limit states in anisotropic elastic solids proposed by Rychlewski [5] for prediction of elastic limit states in cellular materials. The analysis is based on elastic model of a skeleton and an idealized description of topological arrangement of cell structure for cellular materials. The considered unit cells have, respectively, the form of a cube, a cuboid, a simple prism with the base of equilateral triangle, and a simple prism with the base in the form of regular hexagon. The morphology of the skeleton in a particular unit cell modeled by means of the struts joined in a rigid node determines the elastic stiffness and its symmetry: cubic symmetry, orthotropy and transversal symmetry. An analytical formulation of force-displacement relations for the skeleton struts is found by considering the affinity of node displacements in tensile, bending, and shear deformation. The elements of the stiffness matrix for a single cell are expressed as functions of the compliance coefficients for stretching and bending of struts. The analytical formulae for the elastic Kelvin moduli and the critical energy densities as well as the graphical presentation of the results were obtained with application of symbolic operations provided by Mathcad program. The distributions of critical energy density of particular elastic eigen states with respect to the change of the stiffness of the skeleton were studied.

Celem pracy jest zastosowanie energetycznego kryterium stanów granicznych w anizotropowych ciałach sprężystych, które zostało zaproponowane przez Rychlewskiego [5] do określenia sprężystych stanów granicznych w materiałach komórkowych. Podstawę analizy stanowi model sprężystego zachowania się materiałów komórkowych o elementarnej komórce w kształcie sześcianu, prostopadłościanu, pryzmy o podstawie trójkąta równobocznego i sześciokąta foremnego. Przyjęto struktury komórkowe o powtarzającym się regularnym układzie prętów połączonych w sztywnym węźle, które mogą odkształcać się sprężyście pod wpływem sił osiowych lub momentów gnących i sił poprzecznych. Taki układ charakteryzuje się sztywnością, która może determinować sprężyste własności o symetrii kubicznej, ortotropowej lub transwersalnej. Zaproponowano analityczny sposób wyznaczenia gęstości energii granicznych oraz przedstawiono geometryczną reprezentację zgromadzonej energii w poszczególnych stanach własnych przy jednoosiowym rozciąganiu. Wykorzystano przy tym program do obliczeń symbolicznych Mathcad. Przeprowadzono również analizę wpływu sztywności struktury na rozkład gęstości energii granicznych dla poszczególnych stanów własnych.

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1. Introduction

Usually empirical strength hypotheses are used in order to determine the yield or fracture limit in complex states of stress for various advanced materials having e.g. composite, or cellular structure and characterized in general by anisotropic properties. It appears, however, that it is possible to formulate, at least as a starting point, the theory of strength (material effort) of anisotropic materials if we confine to the analysis of elastic limit states. In such a case the theory is based on the assumption that the elastic energy density is a measure of material effort. It is worthwhile to note that early works of M.T. Huber [1], W.T. Burzyński [2], [3] and more recently by J. Rychlewski [4-6], as well as J. Ostrowska-Maciejewska and J. Rychlewski [7] provided the basis for the formulation of energy-based criteria of elastic limit states. This subject was further studied in [8] and [9]. Also P.S. Theocaris [10] and M.W. Biegler and M.M. Mehrabadi [11] as well as Y.P. Arramon et al. [12] contributed independently to the anisotropic energy-based strength criteria, which are considered separately for particular elastic eigen states.

The aim of the paper is to apply the mentioned above theoretical results of [4-7] to identify and specify the general form of energy-based Rychlewski criterion for different kinds of cellular materials. Following the new idea presented in [13] and [14] about the calculation of the energy limits of elasticity for the pertinent elastic eigen states from first principles (*ab initio* calculations) we propose how to derive the analytical formulae for the critical energy densities accounting for the elastic deformation and the yield strength of the skeleton with its particular morphology and symmetry. Such analytical formulae for the critical elastic energy densities based on elementary interactions in a microstructure were derived for an open-cell foam in [15]. Modelling possibilities of the influence of the strength of struts forming the cellular structure of diverse symmetries on the distribution of energy limits is also studied. We assume that essential macroscopic features of mechanical behaviour of cellular materials can be inferred from the deformation response of a representative microstructural element following the approach presented in the monograph by L.J. Gibson, M.F. Ashby [16] and in [17]. Further references can be also found in [15]. The analysis is based on material properties of a solid phase (skeleton) and topological arrangement of cell structure for a wide range of cellular materials characterized by different types of symmetries, morphologies and type of solid materials from which microstructure is built. The considered unit cells have, respectively, the form of a cube - Fig.1, a cuboid – Fig.2, a simple prism with the base of equilateral triangle – Fig. 3 and a simple prism with the base in the form of regular hexagon – Fig. 4. As it is shown in Figs. 1-4, the morphology of the skeleton in a particular unit cell modeled by means of the struts joined in a rigid node determines the elastic stiffness and its symmetry: cubic symmetry, orthotropy and transversal symmetry. In [15], [18], [19] constitutive description of the linear elastic behaviour of honeycombs and open-cell foams is developed on the basis of microstructural modelling of their skeleton. An analytical formulation of force-displacement relations for the skeleton struts can be found by considering the affinity of node displacements in tensile, bending, and shear deformation. The elements of the stiffness matrix for a single cell are expressed as functions of the compliance coefficients for stretching and bending of struts. The analytical formulae for the elastic Kelvin moduli and the critical energy densities as well as the graphical presentation of the results were obtained with application of symbolic operations provided by Mathcad program [20]. The distributions of critical energy density of particular elastic eigen states with respect to the change of the stiffness of the skeleton were also studied. The presented analysis can be applied for ceramics, polymers as well as for honeycombs and intermetalics having cellular structure on macroscopic level or in micro-scale.

2. Energy-based criterion for cellular materials

Different limit states of cellular materials can be considered, [16]. Each limit state is related with a particular mechanism of failure of the skeleton elements. The compression tests carried out on specimens made of cellular materials reveal the limit states, which show that a linear elasticity range transforms into a range of non-linear elastic behaviour followed usually by an extensive region of permanent strains, produced by buckling or crushing of the skeleton. This leads to localized forms of overall deformation terminated by densification and final squashing of the specimen. An energy approach to the heterogeneous deformation modes in open-cell foams, which is based on the condition of the lack of convexity of the governing energy functional was presented in [21]. The subject of our interest is the state corresponding to the limit of linear elasticity, which corresponds to the onset of yield in the skeleton struts according to the Huber-Mises criterion for isotropic ductile material of the skeleton. If the skeleton is made of a brittle material or a material with different values of strength under tension and compression the other criterion should be applied to calculate the elastic limit states of cellular materials, e.g. Burzyński criterion [3], [4]. For such a definition of the limit state we can formulate precisely a measure of material effort as the density of elastic energy corresponding to a particular elastic eigen state, which can be determined by the symmetry of the limit tensor $\mathbf{H}$ describing the range of elastic behaviour according to the criterion for anisotropic solids of R. von Mises [22]

$$\boldsymbol{\sigma} \cdot \mathbf{H} \cdot \boldsymbol{\sigma} = H_{ijkl}\sigma_{ij}\sigma_{kl} \leq 1 \tag{1}$$

J. Rychlewski [5], [6] proved that “the Mises limit criterion bounds the weighted sum of stored elastic energies corresponding to uniquely defined, energy orthogonal parts of stress”, [6], p. 169. This means that the Mises limit criterion has the energy interpretation for any anisotropic material:

$$\begin{gathered}\boldsymbol{\sigma} \cdot \mathbf{H} \cdot \boldsymbol{\sigma} = \frac{\Phi(\boldsymbol{\sigma}_1)}{\Phi_1^e} + \cdots + \frac{\Phi(\boldsymbol{\sigma}_p)}{\Phi_p^e} \leq 1, \quad p \leq 6 \\ 2\Phi(\boldsymbol{\sigma}_i) = \boldsymbol{\sigma}_i \cdot \mathbf{C} \cdot \boldsymbol{\sigma}_i = C_{klmn}\sigma_{kl}^{(i)}\sigma_{mn}^{(i)}, \ i = 1,\ldots,p\end{gathered} \tag{2}$$

where $\boldsymbol{\sigma} = \boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2 + \cdots + \boldsymbol{\sigma}_p$ is the exactly one energy orthogonal decomposition of the stress tensor determined by the symmetry of $\mathbf{H}$, which in our case is assumed to be the same as the symmetry of elastic compliance tensor $\mathbf{C}$, $\Phi(\boldsymbol{\sigma}_i) = \dfrac{\boldsymbol{\sigma}_i^2}{2\lambda_i}$ is the elastic energy density stored in the pertinent eigen state i, λ_i denotes the elastic Kelvin modulus in the elastic eigen state i, and Φ_i^e is the energy limit of elasticity in the elastic eigen state i, which according to [9] is called the Rychlewski modulus. If the energy limit of certain eigen state tends to infinity, we say that this state is safe for any state of stress. For example, in the theory of plasticity of isotropic metallic solids it is often assumed that the spherical parts of stress tensor are safe. In the case of cellular materials such an assumption does not hold. Then all limits of elasticity should have finite positive values. The values of energy limits of elasticity (Rychlewski moduli) should be determined experimentally, what in particular for the case of cellular solids revealing elastic anisotropy is not an easy task. Therefore, following the idea presented in [13], [14], we propose to derive the analytical formulae for the Rychlewski moduli accounting for the elastic deformation and the yield strength of the skeleton with its particular morphology and symmetry.

In the case of considered in [18] and [19] cellular structure of a cubic unit cell, Fig.1, the energy-based criterion of elastic limit states (2) holds for $p=3$ and the Rychlewski moduli, Φ_I^e, Φ_{II}^e and Φ_{III}^e denote respectively the energy limits of elasticity for the eigen state I – hydrostatic one, the eigen state II – deviatoric one related with an extension along one of the edges of a cubic cell with simultaneous shortening of the two remaining ones, and the eigen state III – deviatoric one related with the change of the angles between the edges belonging to the same face of a cubic cell.

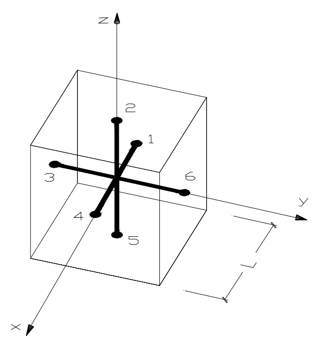

Fig. 1. Cubic cell,
L – strut length, s_n – axial elastic stiffness, s_τ- bending elastic stiffness

For cellular structure of a cuboid unit cell, Fig. 2, the energy-based criterion of elastic limit states (2) holds for $p=6$ and the Rychlewski moduli Φ_I^e, Φ_{II}^e, and Φ_{III}^e denote respectively the energy limits of elasticity for the eigen states: I, II, and III, which are related with the pertinent stretching along the edges of the unit cell, while Φ_{IV}^e, Φ_V^e and Φ_{VI}^e are the energy limits for the eigen states: IV, V, and VI, respectively, related with the change of the angles between the edges belonging to the same face of a cuboid cell.

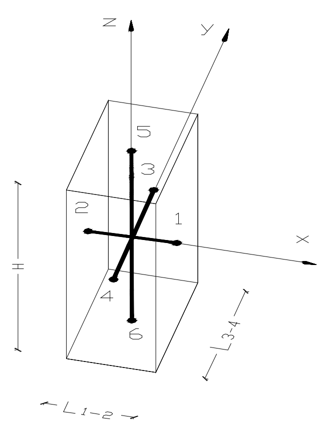

Fig. 2. Cuboid unit cell,
$L_{1\text{-}2}$ - 1-2 strut length, $L_{3\text{-}4}$ - 3-4 strut length, H – 5-6 strut length,
$s_{ni\text{-}j}$ – axial elastic stiffness of strut i-j, $s_{\tau i\text{-}j}$- bending elastic stiffness of strut i-j

For cellular structure of a simple prism with the base of equilateral triangle, Fig. 3, as well as, a simple prism with the base in the form of regular hexagon, Fig. 4, the energy-based criterion of elastic limit states (2) holds for $p=4$ and the Rychlewski moduli Φ_I^e and Φ_{II}^e, denote respectively the energy limits of elasticity for the eigen state I – related with stretching in the base plane along the edges of a unit cell, for the eigen state II – related with stretching along the edges of the unit cell, which are perpendicular to the base, while Φ_{III}^e corresponds to the eigen state III, which can be realized with stretching along one of the edges of the base and simultaneous shearing, and Φ_{IV}^e corresponds to the eigen state IV, which can be realized with the stretching along one of the edges perpendicular to the base and simultaneous shearing.

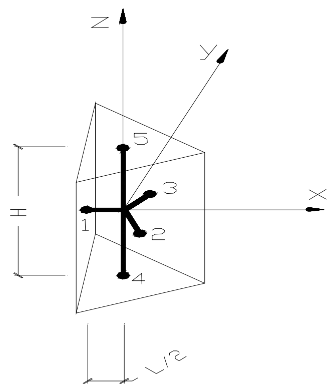

Fig. 3. Simple prism with the base of equilateral triangle,
L - length of struts in XY plane, H – length of struts parallel Z axis,
s_{nL} , s_{nH} – axial elastic stiffnesses, $s_{\tau L}$, $s_{\tau H}$ – bending elastic stiffnesses

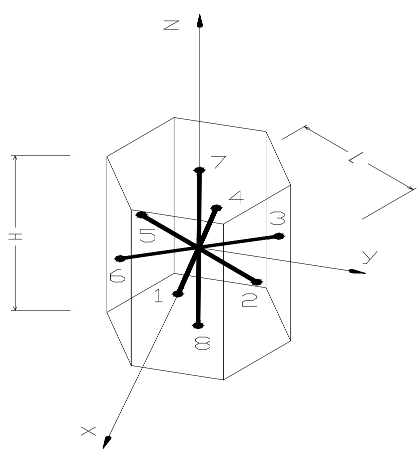

Fig. 4. Simple prism with the base in the form of regular hexagon,
L - length of struts in XY plane, H – length of struts parallel Z axis,
s_{nL} , s_{nH} – axial elastic stiffnesses, $s_{\tau L}$, $s_{\tau H}$ – bending elastic stiffnesses

3. Calculation of the energy limits of elasticity for particular elastic eigen states

The energy limit of elasticity results from analysis of pertinent elastic eigen state for a given unit cell. First the analytical derivation is presented for a cubic unit cell. Knowing the stiffness matrix for a cubic cellular structure [18, 19], three elastic eigen values can be determined, which are known in the literature as Kelvin moduli [6]. For the assumed skeleton geometry and morphology of the cellular structure the Kelvin modulus of the I eigen state – hydrostatic one, belonging to the one-dimensional subspace of six-dimensional stress space, is equal to the elastic modulus of the II eigen state - deviatoric one related with an extension along one of the edges of a cubic cell with simultaneous shortening of the two remaining ones, belonging to the two-dimensional subspace of stress space. The Kelvin modulus of the III eigen state - deviatoric one is related with the change of the angles between the edges of the same face of a cubic cell, belonging to the three-dimensional subspace of stress space. Then the corresponding formulae read as follows

$$\begin{aligned} \lambda_I &= \lambda_1 = \frac{s_n}{2L} \\ \lambda_{II} &= \lambda_2 = \lambda_3 = \frac{s_n}{2L} \\ \lambda_{III} &= \lambda_4 = \lambda_5 = \lambda_6 = \frac{2s_\tau}{4L} \end{aligned} \tag{3}$$

The symbol s_n denotes axial elastic stiffness of the skeleton strut and s_τ is bending stiffness of the skeleton strut regarded as a Timoshenko beam. They are determined by geometrical parameters of the strut: a length L, a cross-section A, and a moment of inertia J of the cross-section, and the elastic constants of the skeleton material: E_s - Young modulus, and G_s - shear modulus. For a prismatic strut the elastic compliance $c_n = s_n^{-1}$ and $c_\tau = s_\tau^{-1}$ is defined respectively as

$$c_n = \frac{L}{2E_S A}, \quad c_\tau = \frac{L^3}{24 E_S J} + \frac{L}{2G_S A_\tau} \tag{4}$$

From the known kinematics of the unit cell we can relate the following components of elastic limit forces in the skeleton struts with the elastic limit state corresponding to a particular elastic eigen state of the cellular material

$$F_n^e = AR_e \tag{5}$$

for tension and compression of a skeleton strut, where R_e is the elasticity limit in tension or compression and

$$F_\tau^e = \frac{R_e I}{Lh} \tag{6}$$

for bending of Timoshenko beam.

Calculation of volume averages of the stress tensor in the skeleton strut $\boldsymbol{\sigma}^S$, components of which taken in the coordinate system of the unit cell are determined by limit forces (5) and (6)

$$\boldsymbol{\sigma} = \frac{1}{V} \int_{V^S} \boldsymbol{\sigma}^S dV \tag{7}$$

where $\boldsymbol{V}$ denotes the unit cell volume, $\boldsymbol{V}^S$ is volume of the skeleton strut, gives the macroscopic stress tensor acting on the faces of unit cell. Projection of the resulting macroscopic stress tensor (7) on the particular elastic eigen states [6] leads to the following formulae for elastic eigen stress in the limit state:

$$\sigma_I = \sigma_1 = \begin{bmatrix} \frac{-A\,R_e}{L^2} & 0 & 0 \\ 0 & \frac{-A\,R_e}{L^2} & 0 \\ 0 & 0 & \frac{-A\,R_e}{L^2} \end{bmatrix}$$

$$\sigma_{II} = \sigma_{2,3} = \begin{bmatrix} \frac{2}{3}\frac{A\,R_e}{L^2} & 0 & 0 \\ 0 & \frac{-1}{3}\frac{A\,R_e}{L^2} & 0 \\ 0 & 0 & \frac{-1}{3}\frac{A\,R_e}{L^2} \end{bmatrix} \tag{8}$$

$$\sigma_{III} = \sigma_{4,5,6} = \begin{bmatrix} 0 & \frac{I\,R_e}{h\,L^3} & \frac{I\,R_e}{h\,L^3} \\ \frac{I\,R_e}{h\,L^3} & 0 & \frac{I\,R_e}{h\,L^3} \\ \frac{I\,R_e}{h\,L^3} & \frac{I\,R_e}{h\,L^3} & 0 \end{bmatrix}$$

In the eigen states II and III the components of stress in elastic limit state were calculated for the simultaneous realization of strain states belonging to their particular subspaces, e.g. trigonal strain for the eigen state III. The corresponding energy limits of elasticity take the form:

$$\begin{aligned} 2\,\Phi_I^e &= \frac{1}{\lambda_I}(\sigma_I \cdot \sigma_I) = \frac{1}{\lambda_I}\,3\left(\frac{A\,R_e}{L^2}\right)^2 \\ 2\,\Phi_{II}^e &= \frac{1}{\lambda_{II}}(\sigma_{II} \cdot \sigma_{II}) = \frac{1}{\lambda_{II}}\,\frac{2}{3}\left(\frac{A\,R_e}{L^2}\right)^2 \\ 2\,\Phi_{III}^e &= \frac{1}{\lambda_{III}}(\sigma_{III} \cdot \sigma_{III}) = \frac{1}{\lambda_{III}}\,6\,\frac{I^2\,R_e^2}{h^2\,L^6} \end{aligned} \tag{9}$$

where h is the maximum distance between the upper and lower fibers of the beam cross-section, A denotes the cross-section area and I is the moment of inertia of the beam cross-section. Performing similar analysis for particular representative cells the analytic relations for Kelvin moduli and elastic energy limits for pertinent eigen states can be obtained.

2. A cuboid unit cell. The eigen values of the stiffness matrix read as follows:

$$\lambda_I = \lambda_1 = \frac{L_{1-2}\ s_{n1-2}}{2\ L_{3-4}\ H}, \qquad \lambda_{II} = \lambda_2 = \frac{L_{3-4}\ s_{n3-4}}{2\ L_{1-2}\ H}, \qquad \lambda_{III} = \lambda_3 = \frac{H\ s_{n5-6}}{2\ L_{1-2}\ L_{3-4}},$$

$$\lambda_{IV} = \lambda_4 = \frac{\dfrac{2\ H^2\ s_{\tau 5-6}}{L_{3-4}{}^2\ s_{\tau 3-4} + H^2\ s_{\tau 5-6}}\ \dfrac{L_{3-4}}{2}\ s_{\tau 3-4}}{L_{1-2}\ H}, \tag{10}$$

$$\lambda_V = \lambda_5 = \frac{\dfrac{2\ H^2\ s_{\tau 5-6}}{L_{1-2}{}^2\ s_{\tau 1-2} + H^2\ s_{\tau 5-6}}\ \dfrac{L_{1-2}}{2}\ s_{\tau 1-2}}{L_{3-4}\ H},$$

$$\lambda_{VI} = \lambda_6 = \frac{\dfrac{2\ L_{3-4}{}^2\ s_{\tau 3-4}}{L_{3-4}{}^2\ s_{\tau 3-4} + L_{1-2}{}^2\ s_{\tau 1-2}}\ \dfrac{L_{1-2}}{2}\ s_{\tau 1-2}}{L_{3-4}\ H}$$

The corresponding energy limits of elasticity in particular eigen states take the form:

$$2\ \Phi_I^e = \frac{1}{\lambda_I}\left(\frac{A\ R_e}{L_{3-4}\ H}\right)^2, \qquad 2\ \Phi_{II}^e = \frac{1}{\lambda_{II}}\left(\frac{A\ R_e}{L_{1-2}\ H}\right)^2, \qquad 2\ \Phi_{III}^e = \frac{1}{\lambda_{III}}\left(\frac{A\ R_e}{L_{1-2}\ L_{3-4}}\right)^2,$$

$$2\ \Phi_{IV}^e = \frac{1}{\lambda_{IV}}\ 8\ \frac{I^2\ R_e^2}{h^2\ L_{1-2}{}^2\ L_{3-4}{}^2\ H^2}, \qquad 2\ \Phi_V^e = \frac{1}{\lambda_V}\ 8\ \frac{I^2\ R_e^2}{h^2\ L_{1-2}{}^2\ L_{3-4}{}^2\ H^2} \tag{11}$$

$$2\ \Phi_{VI}^e = \frac{1}{\lambda_{VI}}\ 8\ \frac{I^2\ R_e^2}{h^2\ L_{1-2}{}^2\ L_{3-4}{}^2\ H^2}$$

3. A simple prism with the base of equilateral triangle. The eigen values of the stiffness matrix read as follows:

$$\lambda_I = \lambda_1 = \frac{\sqrt{3}\ s_{nL}}{6\ H}, \quad \lambda_{II} = \lambda_3 = \frac{2\ \sqrt{3}\ H\ s_{nH}}{9\ L^2},$$

$$\lambda_{III} = \lambda_2 = \lambda_6 = \frac{\sqrt{3}\ s_{nL}\ s_{\tau L}}{3\ H\ (\ s_{nL} + s_{\tau L}\ )}, \quad \lambda_{IV} = \lambda_4 = \lambda_5 = \frac{4\ \sqrt{3}\ H\ s_{\tau H}\ s_{\tau L}}{3\ (\ 3\ L^2\ s_{\tau L} + 4\ H^2\ s_{\tau H}\ )} \tag{12}$$

The corresponding energy limits of elasticity in particular eigen states take the form:

$$2\ \Phi_I^e = \frac{1}{\lambda_I}\ \frac{2}{3}\ \frac{A^2\ R_e^2}{L^2\ H^2}, \qquad 2\ \Phi_{II}^e = \frac{1}{\lambda_{II}}\ \frac{16}{27}\ \frac{A^2\ R_e^2}{L^4},$$

(13)

$$2\,\Phi^{e}_{III}=\frac{1}{\lambda_{III}}\,\frac{2}{9}\,\frac{I^2\;A^2\;R_e^{\,2}\;(61-8\sqrt{3})}{H^2\;L^2\;(4\,I+L\,h\,A)^2}\,,\qquad 2\,\Phi^{e}_{IV}=\frac{1}{\lambda_{IV}}\,\frac{64}{27}\,\frac{I^2\;R_e^2}{H^2\;L^4\;\;h^2}$$

4. A simple prism with the base in the form of regular hexagon. The eigen values of the stiffness matrix read as follows:

$$\lambda_{I}=\lambda_{1}=\frac{\sqrt{3}\;s_{nL}}{2\,H},\quad \lambda_{II}=\lambda_{3}=\frac{\sqrt{3}\;H\;s_{nH}}{3\,L^2}$$

$$\lambda_{III}=\lambda_{2}=\lambda_{6}=\frac{\sqrt{3}\,(\,s_{nL}+2\,s_{\tau L}\,)}{4\,H},\qquad \lambda_{IV}=\lambda_{4}=\lambda_{5}=\frac{2\,\sqrt{3}\;H\;s_{\tau H}\;\,s_{\tau L}}{3\,L^2\;\,s_{\tau L}+2\,H^2\;\,s_{\tau H}} \tag{14}$$

The corresponding energy limits of elasticity in particular eigen states take the form:

$$2\,\Phi^{e}_{I}=\frac{1}{\lambda_{I}}\,6\,\frac{A^2\;R_e^2}{L^2\;H^2}\,,\qquad 2\,\Phi^{e}_{II}=\frac{1}{\lambda_{II}}\,\frac{4}{3}\,\frac{A^2\;R_e^2}{H^2\;L^2}\,,$$

$$2\,\Phi^{e}_{III}=\frac{1}{\lambda_{III}}\,30\,\frac{(s_{nL}+2\,s_{\tau L})^2\;R_e^2\;A^2\;\;I^2}{H^2\;L^2\;(4\,s_{nL}\;I+L\,s_{\tau L}\;h\,A)^2}\,,\qquad 2\,\Phi^{e}_{IV}=\frac{1}{\lambda_{IV}}\,\frac{16}{3}\,\frac{I^2\;R_e^2}{H^2\;L^4\;h^2} \tag{15}$$

4. Elastic energy density distribution to individual eigen states – graphical representation

In [4] the elastic energy density function Φ was depicted geometrically by means of the surface of constant energy $2\Phi(\boldsymbol{\sigma})=1$, which takes form of the six dimensional ellipsoid in the space of symmetric tensors of the second order. The axes of the ellipsoid are directed along the axes of the elastic eigen states. The similar geometrical interpretation has the energy-based criterion (2). In the case of cubic symmetry – a unit cell in the form a cube Fig. 1, there is possible to represent graphically the energy based criterion (2) as the three dimensional ellipsoid having the axes directed along the three elastic eigen states [13]. In order to illustrate graphically the energy-based criterion of elastic limit states, specified for different kinds of cellular structure possessing the lower symmetry, we confine our discussion to the eigen states, which are typical for plane state of stress. The elastic energy density distributions to individual elastic eigen states for uniaxial loadings along a family of directions $\boldsymbol{n}$ are considered. The plane state of stress produced by the loading along the axis $\mathbf{n}=cos\,\alpha\mathbf{e}_1+sin\,\alpha\mathbf{e}_2$, in tension ($\sigma_n>0$) or compression ($\sigma_n<0$), is given by the following representation in the orthonormal basis $(\mathbf{e}_1,\mathbf{e}_2)$ orientated along the chosen axes of the considered unit cells:

$$\boldsymbol{\sigma}(\mathbf{n})\leftrightarrow\sigma_n\begin{bmatrix}cos^2\,\alpha & cos\,\alpha\,sin\,\alpha\\ cos\,\alpha\,sin\,\alpha & sin^2\,\alpha\end{bmatrix} \tag{16}$$

The stress tensor of the plane state of stress $\boldsymbol{\sigma}(\mathbf{n})$ can be decomposed into three elastic eigen states for given symmetry of elastic stiffness tensor and represented in the orthogonal coordinates connected with the normalized eigen state vectors. For the material with a cuboid elementary cell corresponding to orthotropic symmetry the energy-based criterion can be depicted in the coordinates of three elastic eigen states as an ellipsoid. The details of the calculations of the elastic eigen states and the limit states surfaces with assumption of hypothetical elastic properties of the skeleton with use of Mathcad program are presented in [23]. The values of the assumed mechanical properties of the skeleton were based on the studies on cancellous bone presented in [24] and applied in [18]. The results of these calculations for a cuboid unit cell are shown in Fig. 5. In the case of transversal isotropy corresponding to a simple prism with the base of equilateral triangle – Fig. 3 and a simple prism with the base in the form of regular hexagon – Fig 4, the elastic energy corresponding to a plane state of stress cumulates in two elastic eigen states. This is depicted in Fig. 6 in the form of ellipse, which corresponds to the transversal isotropic symmetry of hexagonal prism. The similar graphical representation can be obtained for the prism with the base of equilateral triangle. The straight lines reaching the limit energy surfaces in Figures 5 and 6 correspond to possible experiments of axial load determining the elastic limit.

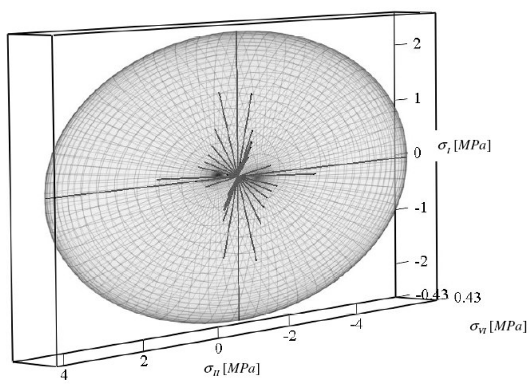

Fig. 5. Visualisation of the stored elastic energy for the eigen states: σ_I – I eigen state, σ_{II} – II eigen state, σ_{VI} – VI eigen state

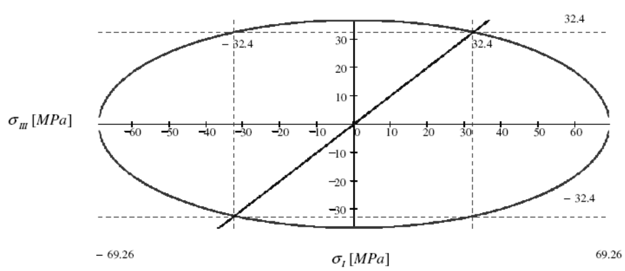

Fig. 6. Visualisation of the stored elastic energy for the eigen states: σ_I – I eigen state, σ_{III} – III eigen state

For the same hypothetical skeleton material the influence of the variation of stiffness of the skeleton struts on the elastic energy density distribution to individual eigen states was studied. The length to diameter ratio of the considered struts was assumed lower than the critical buckling length of the strut under compression. The stiffness of the skeleton was controlled by the change of the struts diameter *d*. The detail analysis with the derivation of analytical formulae and the results of calculations with use of Mathcad program [20] are presented in [25]. Fig. 7 shows an example of the non dimensional elastic energy density distribution to individual eigen states for cellular materials characterising with three typical unit cells: a cubic cell, a cuboid one and a prism with the base of regular hexagon. A simple example of the analytical formulae describing the contributions of the limit energy density for a particular eigen state in the case of the skeleton in the form of a cubic unit cell is given:

$$\frac{\Phi_I^e}{\Phi_I^e+\Phi_{II}^e+\Phi_{III}^e}=\frac{1800}{2275+6k^2+6k^2\nu}, \qquad \frac{\Phi_{II}^e}{\Phi_I^e+\Phi_{II}^e+\Phi_{III}^e}=\frac{400}{2275+12k^2+12k^2\nu},$$

$$\frac{\Phi_{III}^e}{\Phi_I^e+\Phi_{II}^e+\Phi_{III}^e}=\frac{3\left(25+4k^2+4k^2\nu\right)}{2275+12k^2+12k^2\nu}, \qquad k=\frac{d}{d_{max}} \tag{17}$$

where d is the diameter of the strut cross-section and ν denotes the Poisson's ratio of the skeleton material. The more complicated formulae for other unit cells are derived in [25].

a) **b)** **c)**

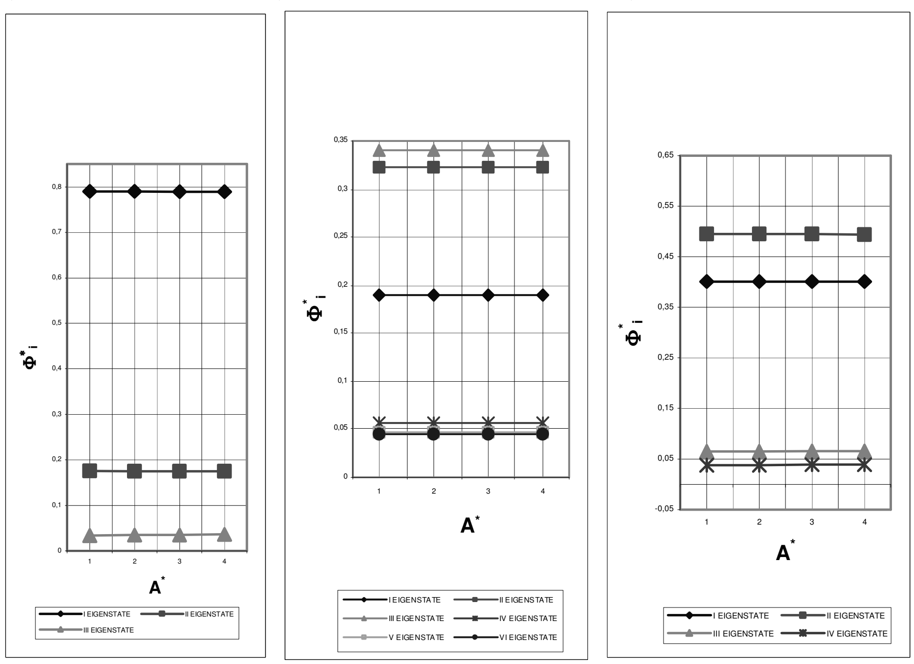

Fig. 7. Disribution of limit energy densites to individual eigen states as a function of the stiffness of the skeleton struts a) – cubic cell, b) – cuboid unit cell, c) - simple prism with the hexagonal base,
A* - non dimensional circular cross – sectional area of the skeleton strut
Φ^*_i- non dimensional limit energy density of the eigen state *i*

5. Conclusions

The paper shows that it is possible for many kinds of cellular materials to obtain analytical formulae describing the elastic limit energy densities in the function of the structural parameters of the skeleton. This opens the possibility, at least in the limits of possible range of structural changes, to tailor the materials according to the assumed functional requirements. The presented analysis can be applied for ceramics, polymers as well as for honeycombs and intermetalics having cellular structure on macroscopic level or in micro-scale.

REFERENCES

1. M.T. Huber, Specific Work of strain as a measure of material effort. A contribution to foundations of the strength of materials theory (in Polish), Czasopismo Techniczne, Lwów XXII, Nr. 3., 38-40, Nr. 4., 49-50, Nr. 5., 61-63, Nr. 6., 80-81 (1904), (English edition in the centennial of Polish publication: Arch. Mech., 56, 173-190 (2004) – translated by A. Stręk under scientific supervision of R.B. Pęcherski).
2. W.T. Burzyński, Theoretical foundations of the hypotheses of material effort (in Polish), Czasopismo Techniczne, Lwów **47**, 1-41, (1929) (English translation of the original paper by A. Stręk under scientific supervision of R.B. Pęcherski) – under preparation.
3. W.T. Burzyński, Ueber die Anstrengungshypothesen, Schweizerische Bauzeitung **94**, 259-262 (1929).
4. J. Rychlewski, on Hooke's law, PMM, **48**, 420-435, (1984), (in Russian); English translation in Prik. Matem. Mekhan., **48**, 303-314, 1984.
5. J. Rychlewski, Elastic energy decomposition and limit criteria, Uspekhi Mekh. - Advances in Mech. **7**, 51-80 (1984), (in Russian).
6. Rychlewski J.: Unconventional approach to linear elasticity, Arch. Mech. **47**, 149-171 (1995).
7. J. Ostrowska-Maciejewska, J. Rychlewski, Plane elastic limit states in anisotropic solids, Arch. Mech. **40**, 379-386 (1988).
8. K. Kowalczyk, J. Ostrowska-Maciejewska, Energy-based limit conditions for transversely isotropic solids, Arch. Mech. **54**, 497-523 (2002).
9. K. Kowalczyk, J. Ostrowska-Maciejewska, R.B. Pęcherski, An-energy based yield criterion for solids of cubic elasticity and orthotropic limit state, Arch. Mech. **55**, 431-448 (2003).
10. P.S. Theocaris, The decomposition of the strain-energy density of a single-ply laminate to orthonormal components, J. Reinf. Plast. Comp. **8**, 565-583 (1989).
11. M.W. Biegler, M.M. Mehrabadi, An energy-based failure criterion for anisotropic solids subject to damage, Damage in Composite Materials, G.Z. Voyiadjis (ed.), Elsevier, 23-34 (1993).
12. Y.P. Aramon, M.M. Mehrabadi, D.W. Martin, S.C. Cowin, A multidimensional anisotropic strength criterion based on Kelvin modes, Int. J. Solids Structures **37**, 2915-2935 (2000).
13. K. Nalepka, R.B. Pęcherski, Physical basis of energy-based criteria of limit states for monocrystals, in: XXX Szkoła Inżynierii Materiałówej, Kraków-Ustroń-Jaszowiec, October 1-4, 2002, (J. Pacyno ed.), AGH, Kraków, 311-316 (2002), (in Polish).
14. K. Nalepka, R.B. Pęcherski, Energy-based criteria of limit states. How to calculate the values of limit energy from first principles, Rudy Metale **48**, 533-536 (2003).
15. Janus-Michalska M., Pęcherski R.B.: Macroscopic properties of open-cell foams based on micromechanical modelling, Technische Mechanik **23**, 234-244 (2003).
16. L.J. Gibson, M.F. Ashby, Cellular Solids: Structure and Properties, Cambridge University Press, 1998.
17. W.E. Warren, K A. M. Kraynik, The linear elastic properties of open foams, J. Appl. Mech. **55**, 341-346 (1988).
18. P. Kordzikowski, M. Janus-Michalska, R.B. Pęcherski, Analysis of the influence of the strength of the struts forming a cubic cell structure on the distribution of the energy limits, Rudy Metale **R49**, No. 3, 216-220 (2004), (in Polish).

19. M. Janus-Michalska, Effective models describing elastic properties of cellular materials – under preparation.
20. Mathcad Professional 2001 Manual.
21. G. Gioia, Y. Wang, A. M. Cuitiño, The energetics of heterogeneous deformation in open-cell solid foams, Proc. R. Soc. Lond., **457A**, 1079-1096 (2001).
22. R. von Mises, Mechanik der plastischen Formänderung von Kristallen, ZAMM **8**, 161-185 (1928) (also in: Selected Papers of Richard von Mises, v. I, ed. by Ph. Frank et al., American Mathematical Society, Providence, 1963, 251-293).
23. P. Kordzikowski, M. Janus-Michalska, R.B. Pęcherski, Visualization of the elastic energy density distributed to the individual eigen states - the internal report No. 5 of the Chair of the Strength of Materials, Institute of Structural Mechanics, CUT, 2004, Kraków (in Polish).
24. P. Kowalczyk, Elastic properties of cancellous bone derived from finite element models of parameterized microstructure cells, J. Biomech., **36**, 961-972, (2003).
25. P. Kordzikowski, M. Janus-Michalska, R.B. Pęcherski, Analysis of the influence of the strength of struts on the distribution of elastic energy limits in cellular materials of different symmetry - the internal report No. 6 of the Chair of the Strength of Materials, Institute of Structural Mechanics, CUT, 2004, Kraków (in Polish).