Elastic Stiffness and Yield Strength of Periodic Closed-Cell Honeycombs

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ABSTRACT: The multiscale modeling idea applied to formulate an effective model of elastic behaviour of closed-cell honeycombs is presented. Essential macroscopic features of mechanical behaviour are inferred from the deformation response of a representative volume element. The structural mechanics methods are applied to the plate model of a skeleton. An analytical formulation of force-displacement relations for the skeleton elements is found by considering the affinity of nodal displacements in tensile, bending and shear deformations. The stiffness tensor for anisotropic solid may be determined depending on material properties of the solid phase and topological arrangement of cellular structure, using micro-macro transition based on micromechanical approach. Macroscopic yield condition is predicted on the basis of energy yield criterion for anisotropic solid. The study based on the assumption of linear elasticity leads to analytical formulae. The proposed theoretical framework may be extended to nonlinear behaviour, plasticity and failure analysis with the use of numerical approach.

1 INTRODUCTION

Cellular materials are made up of interconnected network of cells with solid edges. The regular geometric arrangements of solid skeleton are called honeycombs (Gibson & Ashby (1997)). Such a kind of structure can be found in many natural materials such as wood and cork. A wide range of honeycombs is fabricated using conventional metal bending technology and manufactured from polymers and ceramics. Cellular materials become popular in industrial applications due to their high specific mechanical, thermal, acoustic properties and ultralight weight. Because of their structure natural and synthetic cellular solids exhibit unique physical properties, which support their functionality. The geometrical parameters of the microstructure of a honeycomb can be adapted to the individual needs of various applications. High thermal and acoustic insulating capabilities are associated with closed cell structures. The dominant mechanical properties are stiffness and yield stress. An approach based on multiscale modeling proposed by Philips (2001) and micromechanical analysis by Nemat-Naser & Hori (1999) gives the framework for constructing effective model of any kind of structured material. It was sucessfully adapted for open-cell foams by Janus-Michalska & Pęcherski (2003) and open-cell cellular structures with different topology by Janus-Michalska (2005). The modeling of microstructure with the help of the linear elasticity theory enables to predict the macroscopic yield condition. The general formulation of energy based yield criterion for anisotropic solids was proposed by Rychlewski (1995). The present formulation leads to specification of energy-based limit condition for closed-cell microstructures, taking into account the elementary interactions within the microstructure. This idea was sucessfully applied to metallic foam by Janus-Michalska & Pęcherski (2003) and cellular materials with cubic cells by Kordzikowski & Janus-Michalska & Pęcherski (2004) and to open-cell materials of different symmetries by Janus-Michalska & Kordzikowski & Pęcherski (2005).

2 MICROMECHANICAL ANALYSIS

2.1 *Representative Volume Element*

The considered closed cell material consists of interconnected plate members forming periodic structure as shown in Figure 1. The joints of structural members are treated as fixed. The structure can be modeled using representative unit cell (in micromechanics called RVE) given in Figure 2., consisting of three walls and horizontal plate and having geometrical parameters: h_w, h_p, L, H and material constants of skeleton structure: E_s, ν_s.

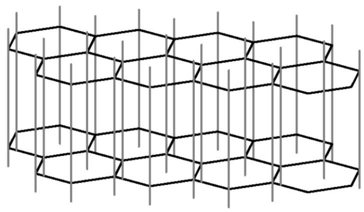

Figure 1. Periodic structure

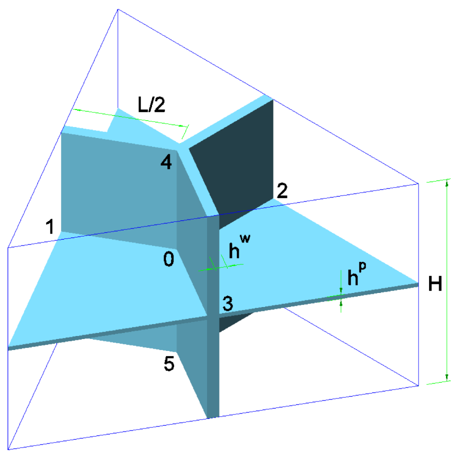

Figure 2. Representative volume element

Geometry of RVE is described by:

$i = 1, ...5$ cell face midpoints,

$\mathbf{b}_i^0 \;\; i = 1, ...5$ midpoint position vectors,

$\mathbf{e}_i^0 \;\; i = 1, ...5$ midpoint position vectors,

h_w, h_p, L, H geometrical parameters of microstructure,

V - cell volume.

2.2 *Strain and stress definition*

The overall response of a periodically structured solid is defined in terms of the volume averages of the stress and strain taken over a typical unit cell:

$$\boldsymbol{\varepsilon} = \left\langle \boldsymbol{\varepsilon}^s \right\rangle_V = \frac{1}{V} \sum_{A_i} sym\left(\mathbf{e}_i \otimes \mathbf{u}_i\right) \, dS$$

$$\boldsymbol{\sigma} = \left\langle \boldsymbol{\sigma}^s \right\rangle_V = \frac{1}{V} \sum_{A_i} \left(\mathbf{t}_i \otimes \mathbf{e}_i\right) \, dS \qquad (1)$$

where: $\langle \; \rangle_V$ stands for the volumetric average in skeleton s taken over V, $\mathbf{e}_i$ is the outer unit normal on the boundary A_i and $\mathbf{u}_i$ and $\mathbf{t}_i$ denote respectively the midpoint displacement on the surface A_i, and surface traction defined as follows: $t_i = F_i / A_i$, where $\mathbf{F}_i$ is an equivalent of uniform load applied over the plate boundary and reduced to point i.

The linear analysis is based on the assumption of the infinitesimal displacements and uniform strains. The affinity of midpoint displacements in uniform strain state is a consequence of periodic structure. In plate model it is assumed that deformation of RVE consists of uniform deformations of plate elements and the displacements compatibility condition is fulfilled only at midpoints. The forces are generated in the microstructure only due to relative components of midpoint displacements Δ_{0-i} with respect to rigid motion of 0 node. The rigid motion of 0 (central) node is determined by the equilibrium condition of unit cell.

2.3 *Plate model of the skeleton structure*

Three independent uniform deformations of structural members shown in Figure 3 are considered: axial extension, bending and shear. The elastic behaviour of plate subject to uniform axial and transverse bending and shearing loads is known from classical solutions. The proper differential equations with appropriate boundary conditions may be found for instance in Wang (2000).

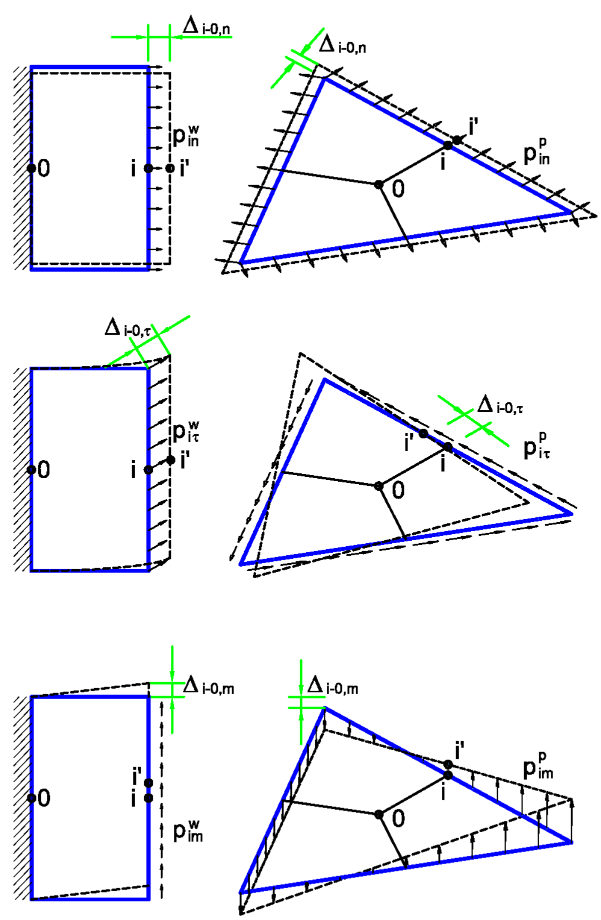

Figure 3. Uniform deformations of structural elements: axial extension, bending and shear.

For each deformation displacement Δ_{i-0} should be compatible for wall and horizontal plate. It is the condition which should be fulfilled for micromechanical model.

The first deformation is produced by uniform axial load p_{in}^{w} for wall elements and p_{in}^{p} for horizontal plate element, which gives equivalent axial forces F_{in}, F_{kn}. The walls bending is related to horizontal plate shear and is produced by $p_{i\tau}^{w}$, $p_{i\tau}^{p}$ respectively, which reduced to i midpoint give $F_{i\tau}$ forces. The walls shear is related to plate bending and is produced by p_{im}^{w}, p_{im}^{p} ,which give F_{im} forces.

We define microstructural stiffnesses as following expressions of geometrical and material parameters:

$$s_{nnL} = \frac{2E_s}{1-\nu_s^2}\left(\frac{H}{L}h_w + \sqrt{3}h_p\right), \qquad s_{nmL} = \frac{2E_s\nu_s}{1-\nu_s^2}h_w,$$

$$s_{\tau\tau L} = 2E_s\left(\frac{H}{L^3}h_w^3 + \sqrt{3}h_p\right), \qquad s_{nmH} = \frac{3E_s\nu_s}{1-\nu_s^2}h_w,$$

$$s_{nnH} = \frac{3E_s}{1-\nu_s^2}\left(\frac{h_w L}{H}\right), \qquad s_{mmH} = \frac{E_s h_w}{1+\nu_s}. \tag{2}$$

These stiffnesses are coefficients in linear relation between the displacements and forces. The relation for given structure has the following form:

$$F_{in} = \Delta_{0-i,n}s_{nnL} + \sum_{j=4}^{5}\Delta_{0-j,m}s_{nmL}, \qquad F_{i\tau} = \Delta_{0-i,\tau}s_{i\tau\tau L},$$

$$F_{im} = \Delta_{0-i,m}s_{mmL} + \sum_{j=4}^{5}\Delta_{0-j,n}s_{nmH}, \qquad i=1,2,3.$$

$$F_{kn} = \Delta_{0-k,n}s_{nnH} + \sum_{j=1}^{3}\Delta_{0-j,n}s_{nnL}, \qquad F_{k\tau} = \Delta_{0-k,\tau}s_{k\tau\tau H},$$

$$F_{km} = \Delta_{0-k,m}s_{mmH} + \sum_{j=1}^{3}\Delta_{0-j,n}s_{nmL}, \qquad k=4,5. \tag{3}$$

where:

n represents direction normal to surface A_i ,

τ direction tangent and perpendicular to the i-th plate member (wall) ,

m direction tangent and paralell to the i-th plate member (wall).

2.4 *Effective elasticity tensor*

Six types of specific deformations of the whole microstructure related to subsequent strain tensor components being non zero one at a time are considered. The forces obtained using the formula (2) make the calculation of stresses in an equivalent continuum possible following the definition (1). The effective stiffness matrix $\mathbf{S}$ which define the linear relation $\boldsymbol{\sigma} = \mathbf{S}\boldsymbol{\varepsilon}$ for equivalent continuum is constructed as the result of such analysis.

For the given cell type the elasticity tensors exhibits transversely isotropic symmetry. Its nonzero components are given below. Generally these components are functions of structural element stiffnesses and geometrical parameters of representative volume elements.

$$S_{1111} = \frac{\sqrt{3}}{12H}\left(\frac{s_{nnL}+3s_{\tau\tau L}}{s_{nnL}+s_{\tau\tau L}}\right)s_{nnL}, \qquad S_{3333} = \frac{2\sqrt{3}}{9L^2}Hs_{nnH},$$

$$S_{1122} = \frac{\sqrt{3}}{12H}\left(\frac{s_{nnL}-s_{\tau\tau L}}{s_{nnL}+s_{\tau\tau L}}\right)s_{nnL}, \qquad S_{1133} = \frac{\sqrt{3}}{3L}s_{nmL}, \tag{4}$$

$$S_{2233} = \frac{\sqrt{3}}{3L}s_{nmL}, \qquad S_{2323} = \frac{2\sqrt{3}}{3}\left(\frac{s_{nmH}}{3Ls_{nmL}+4Hs_{nmH}}\right)s_{nmL}.$$

2.5 *Effective material constants*

Young's modulus is defined as the ratio of tensile stress to tensile strain and can be obtained using the following formula:

$$\frac{1}{E(\mathbf{n})} = (\mathbf{n}\otimes\mathbf{n})\cdot\mathbf{C}\cdot(\mathbf{n}\otimes\mathbf{n}) \tag{5}$$

where: $\mathbf{n}$ normalized vector specifying the tensile direction in a tension test.

Dimensionless spherical diagrams of Young's moduli for structures exhibiting the geometrical properties presented in Table 1., are shown in Figure 4.

Choosen material constants such as: maximum value of Young's modulus $E_{\max}$, shear modulus in horizontal plane G_1 and Poisson's ratio ν_1 and shear modulus in vertical plane G_2 for the considered microstructures and isotropic skeleton material (aluminium) with constants: $E_s = 70\ MPa$, $\nu_s = 0.3$, are given in Table 2.

Table 1. Geometrical parameters of microstructures

	L [m]	H [m]	h^w [m]	h^p [m]
structure a)*	$3\cdot10^{-3}$	$6\cdot10^{-3}$	$1.0\cdot10^{-3}$	$0\cdot10^{-3}$
structure b)	$3\cdot10^{-3}$	$10\cdot10^{-3}$	$0.5\cdot10^{-3}$	$0.2\cdot10^{-3}$
structure c)	$9\cdot10^{-3}$	$6\cdot10^{-3}$	$0.5\cdot10^{-3}$	$0.2\cdot10^{-3}$
structure d)	$18\cdot10^{-3}$	$6\cdot10^{-3}$	$0.5\cdot10^{-3}$	$0.2\cdot10^{-3}$

*a) consists only of walls

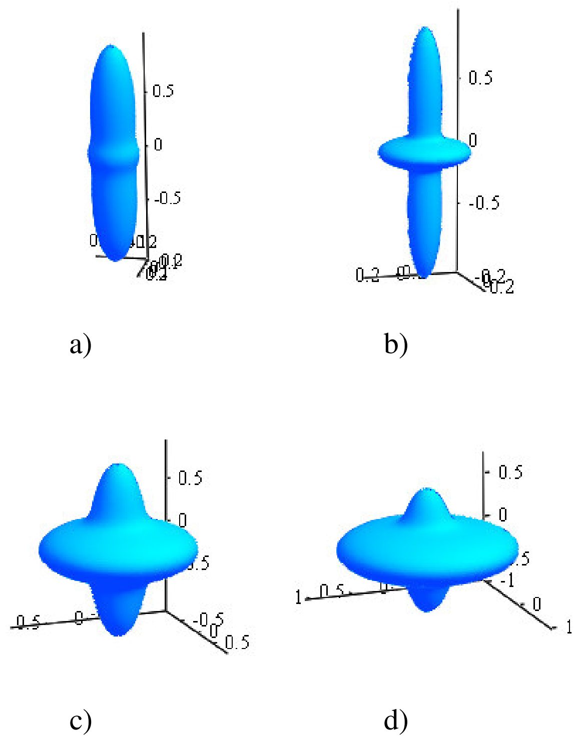

Figure 4. Graphical representation of Young's modulus.

Table 2. Material constants

	$E_{\max}$ [MPa]	G_1 [MPa]	G_2 [MPa]	ν_1
structure a)	$1.895\cdot10^4$	$1.359\cdot10^3$	$1.776\cdot10^3$	0.554
structure b)	$1.039\cdot10^4$	$1.348\cdot10^3$	$0.579\cdot10^3$	0.399
structure c)	$4.063\cdot10^3$	$1.598\cdot10^3$	$0.635\cdot10^3$	0.129
structure d)	$3.105\cdot10^3$	$1.446\cdot10^3$	$0.444\cdot10^3$	0.074

3 YIELD CRITERION

3.1 *Micro-macro transition*

Definition of equivalent stress tensor gives relation between stress in skeleton material and macrostress. When the microscopic level is considered, the HMH hypothesis is adopted for isotropic skeleton material. Stress on macroscale is critical when elastic distortion energy density reaches critical value for a skeleton point. Presented formulation can give theoretically determined critical stress for equivalent continuum derived from interactions in microstructure. Since the analytical formula generated by Mathcad for even the simplest case of tension is a very long expression, it is presented in shortened form, as function of geometric parameters, elasticity limit of skeleton material and its elastic constants and direction of tension:

$$\sigma_{CR} = \sigma_{CR}\left(L, H, h_w, h_p, R_e^s, E_s, \nu_s, \alpha\right) \tag{6}$$

It may be interpreted as a prediction of critical stress depending on microstructural parameters.

4 CONCLUSION

The main advantage of such an approach is that the macroscopic constitutive model follows readily from the analytical treatment. The elastic stiffness and initial strength are expressed in terms of geometric and material parameters of skeleton structure. Thus it is possible to tailor the material to special mechanical requirements in the elastic range. Since this model based on micromechanics is simplified, numerical and experimental verification is required before application.

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