MICROMECHANICAL MODEL OF HYPERELASTIC BEHAVIOUR OF CELLULAR MATERIALS

Małgorzata Janus-Michalska

Solmech 2008, Gdańsk 9-12.09.2008

CONTENTS

1. Small strain anisotropic hyperelasticity

- strain measures
- stress measures
- constitutive equation

2. Micromechanical analysis

- microstructural deformation
- mechanical model of cellular skeleton structure
- initial stiffness matrix
- evaluation of elastic range

3. Uniform deformations-results

- uniaxial tension-compression
- biaxial tension-compression
- simple shear test

4. Conclusions

5. Literature

SMALL STRAIN ANISOTROPIC ELASTICITY

Stress measures

$\mathbf{T}^{ij}$ - first Piola-Kirchhoff stress tensor

directions of the initial tangent base vectors in the undeformed state

(nominal stress)

Π^{ij} - second Piola-Kirchoff stress tensor

directions of tangent base vectors in the deformed state

relation:

$$\mathbf{\Pi} = \mathbf{F}^{-1}\mathbf{T}$$

Strain measure physical Green's Lagrangian strain tensor

definition of components:

$$\mathrm{E_{ii}} := \frac{1}{2}\left(\Lambda_i^2 - 1\right)$$

where:

$\Lambda_i = \dfrac{dX_i}{dx_i}$ - relative stretch in the direction of *i* axis

$$\mathrm{E_{ij}} := \frac{1}{2}\Lambda_i \Lambda_j \sin\Gamma_{ij}$$

where:

Γ_{ij} - change of angle between two different *ij* axes

deformation is given by deformation gradient **F** $\Rightarrow \quad 2\mathbf{E} = \mathbf{F}\mathbf{F}^{\mathrm{T}} - \mathbf{I}$

Constitutive equations

neo-Hookean constitutive relationship
between Green's strain tensor and the second Piola-Kirchhoff stress tensor

the case of small strains :

$$\mathbf{\Pi}\left(\mathbf{E}\right) = \left.\frac{\partial \mathbf{\Pi}}{\partial \mathbf{E}}\right|_{\mathbf{E}=0} : \mathbf{E} = {}^{0}\mathbf{S} : \mathbf{E} \qquad ( )$$

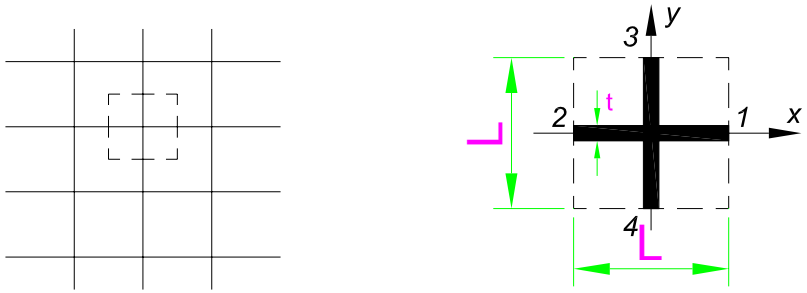
where:

${}^{0}\mathbf{S}$ - initial elasticity tensor (initial tangent operator)

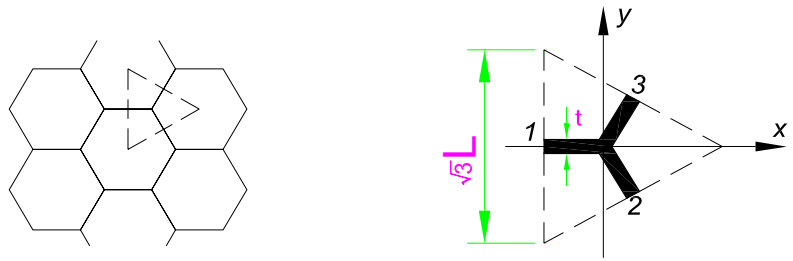
MICROMECHANICAL ANALYSIS

Cellular materials

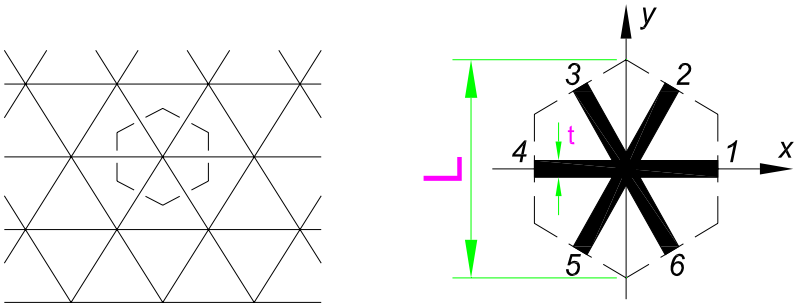
a) square cell structure, square unit cell



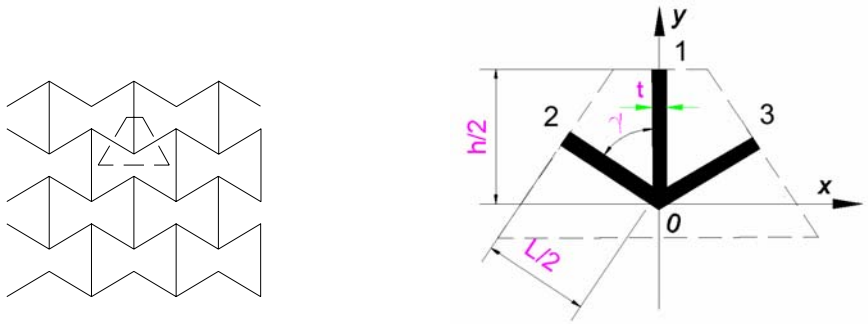
b) ‘honeycomb’ structure, triangle unit cell



c) equilateral triangular structure, hexagonal unit cell

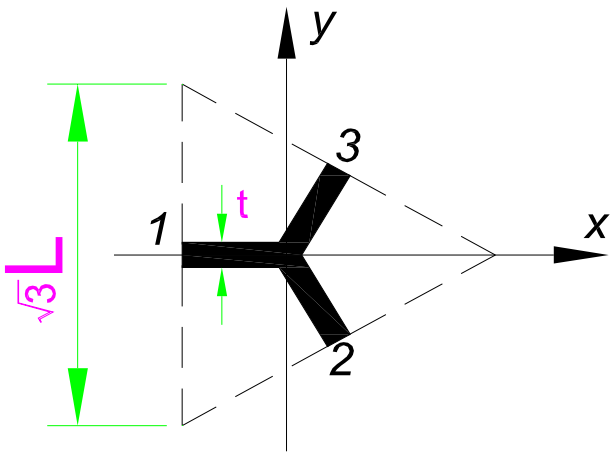


d) ‘reentrant’ structure, trapezoid unit cell



Typical plane structures and representative unit cells

Microstructural deformation - kinematics



$$\boldsymbol{\varepsilon} = \frac{1}{V} \sum_{A_i} sym\left(\mathbf{n}_i \otimes \Delta_i\right)\, dS$$

where: Δ_i - midpoint displacement vector,

$\mathbf{n}_i$ - unit normal to the cell boundary.

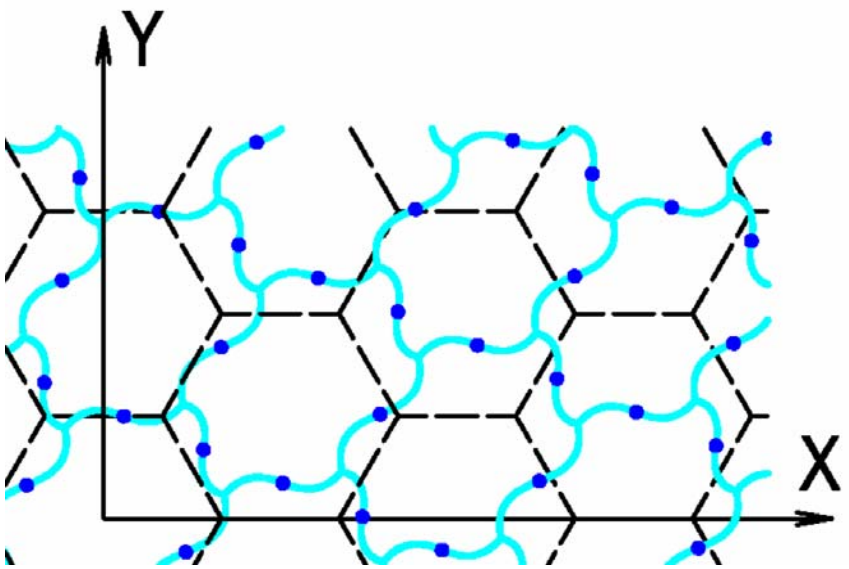
$\boldsymbol{V}$ - volume of representative unit cell

A_i - face areas perpendicular to struts i.

notation: $\mathbf{b}_i^0$ - position vectors, $\mathbf{n}_i = \mathbf{b}_i^0 / \left|\mathbf{b}_i^0\right|$ $\quad i = 1, 2, ..n$, where: $\left|\mathbf{b}_i^0\right| = L_{0-i} / 2$.

unit **uniform strain field** on the unit cell ${}^{K}\tilde{\varepsilon}$, $K = 1,2,3$,

$\mathbf{\Delta}_i = \mathbf{\Delta}_i\left({}^{K}\tilde{\varepsilon}\right)$ $K = 1,2,3$ the displacements for the midpoints $i = 1,2,..n$



$$\mathbf{\Delta}_i = \mathbf{\Delta}_{i-0} + \mathbf{\Delta}_0 + \boldsymbol{\psi} \times \mathbf{b}_0 \qquad i = 1,2,..n$$

where:

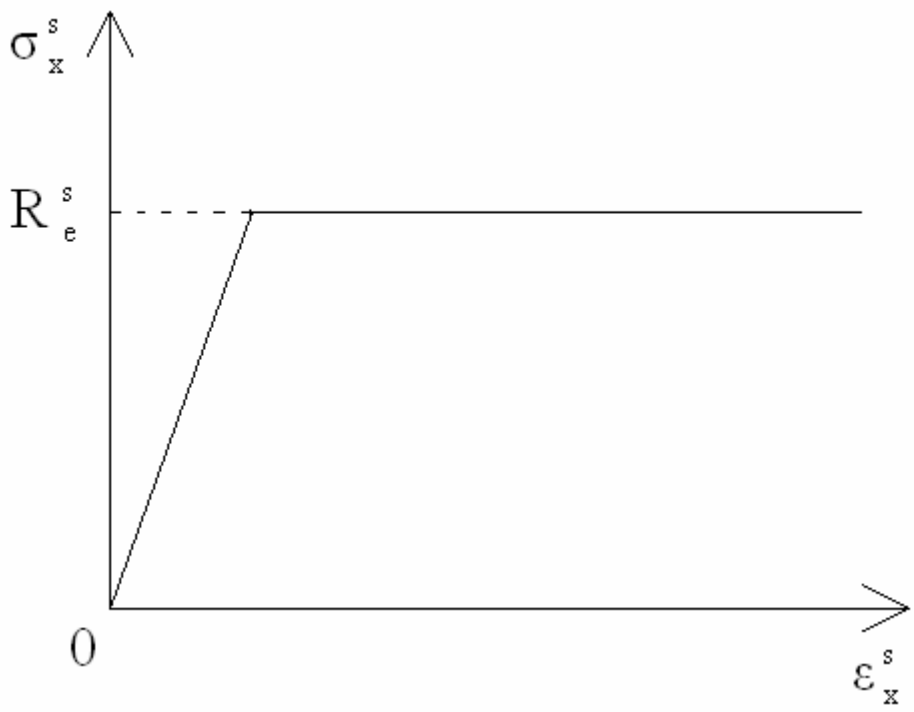
$\mathbf{\Delta}_0$ - translational component, $\boldsymbol{\psi}$ - rotation (effect of reorientation)

$\mathbf{\Delta}_{0-i}$ - relative midpoint displacement with respect to node

Mechanical model of cellular skeleton structure

Skeleton material:

E_s - Young's modulus, ν_s - Poisson's ratio, R_e^s - Yield stress



Timoshenko beam model

resultant forces for the sets of midpoint displacements $\mathbf{\Delta}_i = \mathbf{\Delta}_i\left({}^K\tilde{\varepsilon}\right) \qquad K = 1,2,3$

${}^K\tilde{F}_{ni}\left(\xi_i\right)$ axial force,

${}^K\tilde{F}_{\tau i}\left(\xi_i\right)$ transversal force

${}^K\tilde{M}_i\left(\xi_i\right)$ bending moment functions

where:

ξ_i - local coordinate axis for strut i, $\xi_i\left(0\right) = 0,\ 0 \le \xi_i \le \frac{L_{0-i}}{2}$, i=1,2,...n.

obtained using FEM code.

a, b, c) structure analytical solutions also obtained in Mathcad program.

Microscale

$\boldsymbol{\sigma}^s$ - stress tensor in each point of beam skeleton structure

Initial stiffness matrix

averaging the strain energy density:

$$\Phi_E = \left\langle {}^s\Phi_E \right\rangle_V = \frac{1}{V}\int_{V_s} \left({}^s\Phi_E \right) dV_s$$

where:

$\langle \ \rangle_V$ - volumetric average in skeleton s taken over V

V- volume of unit cell,

Vs- volume of skeleton in unit cell

stiffness matrix components for equivalent continuum

$$ {}^{0}S_{IJ} = \frac{1}{V}\left(\frac{\partial^2 \int_{V_s} \left({}^{s}\Phi_E dV_s \right)}{\partial\left({}^{I}\varepsilon \right)\ \partial\left({}^{J}\varepsilon \right)} \right) $$

Kelvin notation (in 6-D space) - tensor representation is 3×3 matrix.

structures a, b, c, - analytical formulae

$$ S_{jklm} = S_{jklm}\left(L_{o-1}, ..L_{o-i}, ...L_{o-n}, t,\ \gamma,\ E_s, \nu_s \right) $$

structure d - components obtained numerically

a) $$\mathbf{S}=\begin{bmatrix} \frac{E_s t}{L} & 0 & 0 \\ 0 & \frac{E_s t}{L} & 0 \\ 0 & 0 & \frac{E_s t^3}{L^3} \end{bmatrix}$$

b) $$\mathbf{S}=\begin{bmatrix} \frac{\sqrt{3}E_s t}{6L}\frac{\left(L^2+3t^2\right)}{\left(L^2+t^2\right)} & \frac{\sqrt{3}E_s t}{6L}\frac{\left(L^2-t^2\right)}{\left(L^2+t^2\right)} & 0 \\ \frac{\sqrt{3}E_s t}{6L}\frac{\left(L^2-t^2\right)}{\left(L^2+t^2\right)} & \frac{\sqrt{3}E_s t}{6L}\frac{\left(L^2+3t^2\right)}{\left(L^2+t^2\right)} & 0 \\ 0 & 0 & \frac{\sqrt{3}E_s t^3}{3L\left(L^2+t^2\right)} \end{bmatrix}$$

c) $$\mathbf{S}=\begin{bmatrix} \frac{\sqrt{3}E_s t\left(3L^2+2t^2\right)}{4L^3} & \frac{\sqrt{3}E_s t\left(L^2-t^2\right)}{4L^3} & 0 \\ \frac{\sqrt{3}E_s t\left(L^2-t^2\right)}{4L^3} & \frac{\sqrt{3}E_s t\left(3L^2+2t^2\right)}{4L^3} & 0 \\ 0 & 0 & \frac{\sqrt{3}E_s t}{4L} \end{bmatrix}$$

d) $$\mathbf{s}=\begin{bmatrix} s_{11} & s_{12} & 0 \\ s_{12} & s_{22} & 0 \\ 0 & 0 & s_{33} \end{bmatrix}$$

Evaluation of elastic range

energy-based limit condition (Rychlewski) for anisotropic solids on a macroscale

$$\sum_{\alpha=\mathrm{I}}^{\mathrm{III}} \frac{{}^{\alpha}\Phi_E}{{}^{\alpha}\Phi_E^{\mathrm{cr}}} = 1 \qquad)$$

where:

${}^{\alpha}\Phi_E^{\mathrm{cr}}$ - critical energy density stored in an α eigenstate $\alpha = \mathrm{I,II,III}$.

${}^{\alpha}\Phi_E$ - energy density for arbitrary given strain state stored in alfa eigenstate

geometric nonlinarity:

$${}^{\alpha}\Phi_E^{\mathrm{cr}} = \lambda_\alpha \left({}^{\alpha}\mathbf{E}^{cr} \right)^2, \qquad {}^{\alpha}\Phi_E = \lambda_\alpha \left({}^{\alpha}\mathbf{E} \right)^2$$

where:

λ_α - α eigenvalue of stiffness matrix,

${}^{\alpha}\mathbf{E}$ -α strain eigenstate ,

The calculation algorithm for critical energies.

Macroscopic level uniform deformations

$$^{\mathrm{I}}\tilde{\mathbf{E}} = \left({}^{\mathrm{I}}\tilde{\mathbf{E}}_x, {}^{\mathrm{I}}\tilde{\mathbf{E}}_y, {}^{\mathrm{I}}\tilde{\mathbf{E}}_{xy} \right) \quad , \quad {}^{\mathrm{II}}\tilde{\mathbf{E}} = \left({}^{\mathrm{II}}\tilde{\mathbf{E}}_x, {}^{\mathrm{II}}\tilde{\mathbf{E}}_y, {}^{\mathrm{II}}\tilde{\mathbf{E}}_{xy} \right), \quad {}^{\mathrm{III}}\tilde{\mathbf{E}} = \left({}^{\mathrm{III}}\tilde{\mathbf{E}}_x, {}^{\mathrm{III}}\tilde{\mathbf{E}}_y, {}^{\mathrm{III}}\tilde{\mathbf{E}}_{xy} \right)$$

Microscopic level

forces obtained from numerical solution for unit cell skeleton structure:

$$^{\alpha}\tilde{F}_{in}, \qquad {}^{\alpha}\tilde{F}_{i\tau}, \qquad \alpha = \mathrm{I,II,III}.$$

limit condition for skeleton structure :

$$\max_i \left({}^{\alpha}\sigma_x^s \right) = {}^{\alpha}k \; {}^{\alpha}\tilde{\sigma}_x^s = {}^{\alpha}k \max_i \left(\left| \frac{{}^{\alpha}\tilde{F}_{in}}{A} \right| + \left| \frac{{}^{\alpha}\tilde{F}_{i\tau}}{J} \frac{l_i}{2} \frac{t}{2} \right| \right) = R_e^s \qquad \alpha = \mathrm{I,II,III} \qquad i = 1, 2, ..n$$

scalar multipliers and critical eigenstrains defined :

$$^{\alpha}k := \frac{R_e^s}{{}^{\alpha}\tilde{\sigma}_x^s} \qquad\qquad {}^{\alpha}\mathbf{E}^{cr} = {}^{\alpha}k \cdot {}^{\alpha}\tilde{\mathbf{E}}$$

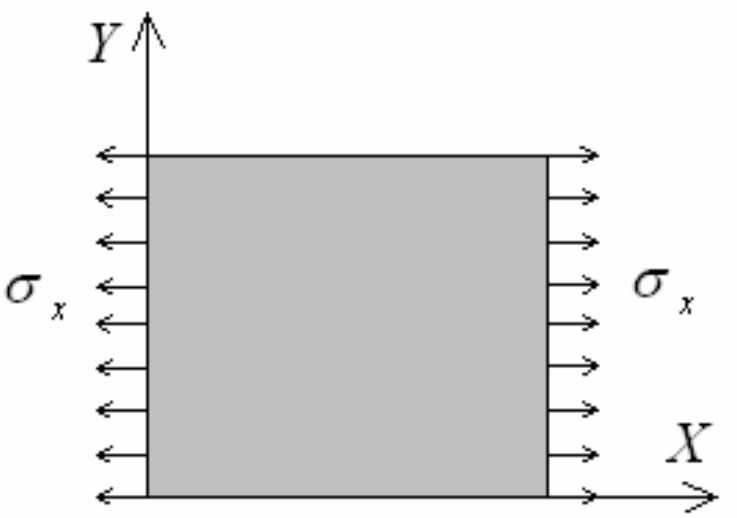
Homogeneous deformations

A) uniaxial load

in tensial and compressive range

$$ {}^{A}\mathbf{F} = \begin{bmatrix} {}^{A}\Lambda_x & 0 \\ 0 & {}^{A}\Lambda_y \end{bmatrix} $$

where: $$ {}^{A}\Lambda_y = \sqrt{\frac{s_{12} + s_{22} - s_{12}\left({}^{A}\Lambda_x\right)^2}{s_{22}}} $$

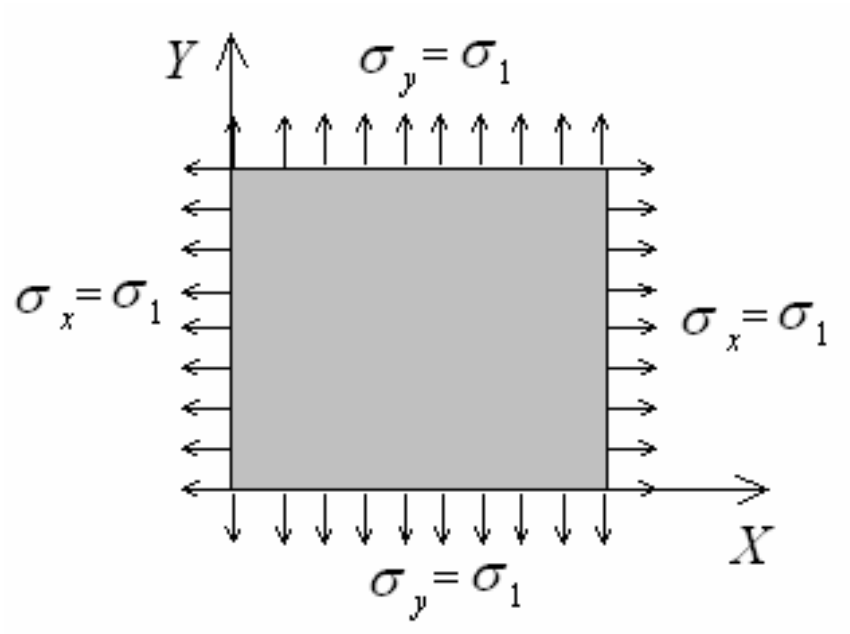


B) biaxial load

in tensial and compressive range

$$ {}^{B}\mathbf{F} = \begin{bmatrix} {}^{B}\Lambda_x & 0 \\ 0 & {}^{B}\Lambda_y \end{bmatrix} $$

where: $$ {}^{B}\Lambda_y = \sqrt{\frac{s_{22} - s_{11} + (s_{11} - s_{12})\left({}^{B}\Lambda_x\right)^2}{s_{22} - s_{12}}} $$



C) simple shear deformation

$$^{C}\mathbf{F} = \begin{bmatrix} 0 & \Gamma \\ 1 & 0 \end{bmatrix}$$

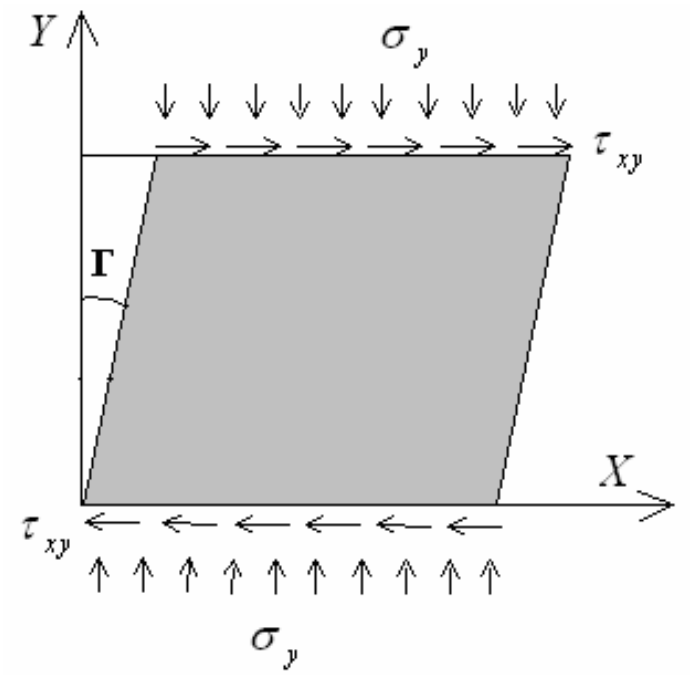
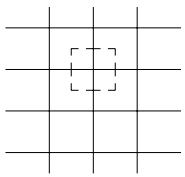
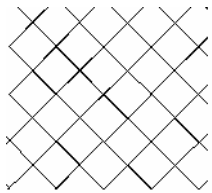
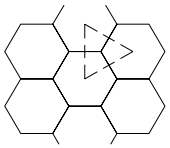


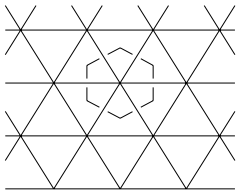
Table 1. Specification of microstructure

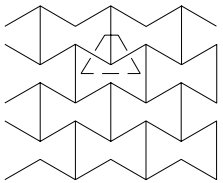
type	Geometric parameters of skeleton [mm]	Skeleton material parameters
a)	$L_{01}=L_{02}=L_{03}=L_{04}=20$, $t=2.0$	$E_S=2GPa$, $v_S=0.33$, $R_e=200$ MPa
b)	$L_{01}=L_{02}=L_{03}=L_{04}=L_{05}=L_{06}=20$, $t=2.0$	$E_S=2GPa$,$v_S=0.33$, $R_e=200$ MPa
c)	$L_{01}=L_{02}=L_{03}=20$, $t=2.0$	$E_S=2GPa$,$v_S=0.33$, $R_e=200$ MPa
d1)	$L_{01}=L_{02}=1.5$, $L_{03}=1.5$, $t=0.15$, $\gamma=80^0$	$E_S=10GPa$, $v_S=0.3$, $R_e=25$ MPa
d2)	$L_{01}=L_{02}=1.575$, $L_{03}=1.575$, $t=0.15$, $\gamma=70^0$	$E_S=10GPa$, $v_S=0.3$, $R_e=25$ MPa
d3)	$L_{01}=L_{02}=1.5$, $L_{03}=3.0$, $t=0.15$, $\gamma=60^0$	$E_S=10GPa$, $v_S=0.3$, $R_e=25$ MPa

a) 

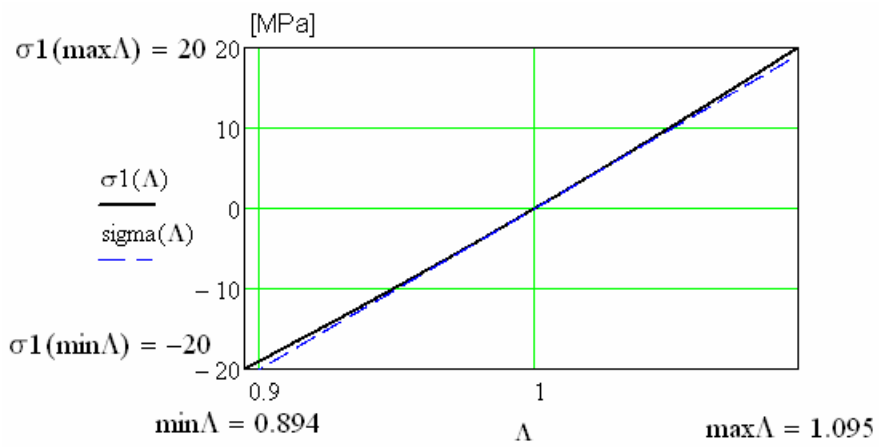
a 45) 

b) 

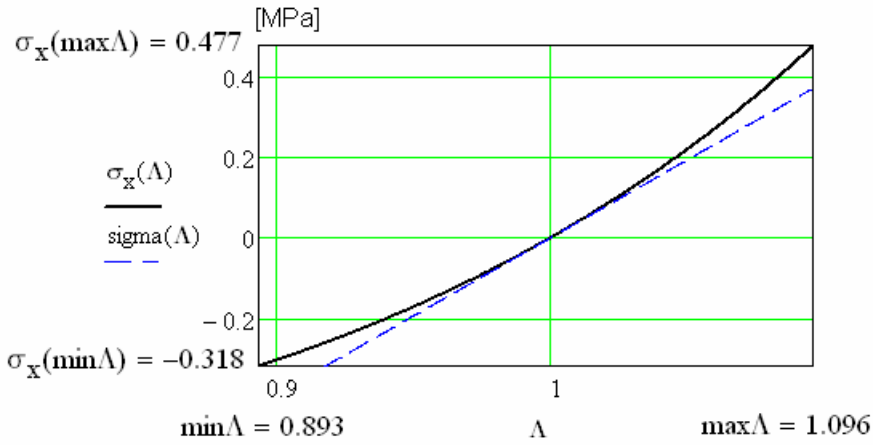
c) 

d) 

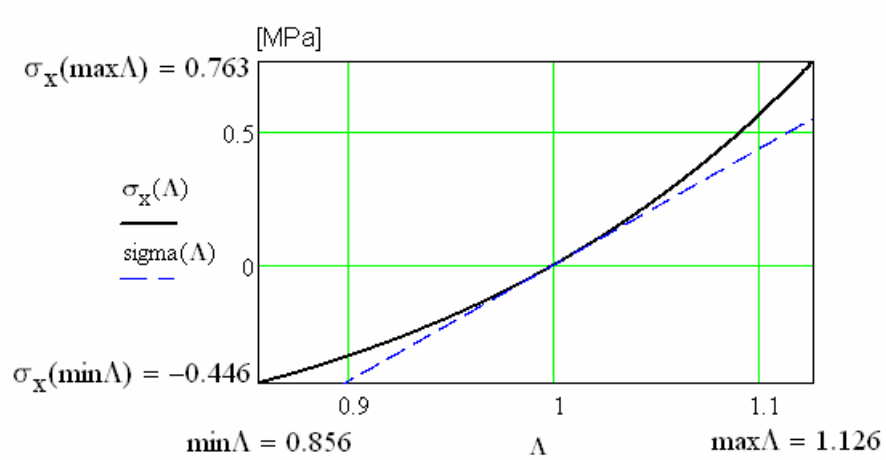
Results A) Uniaxial tension-compression test



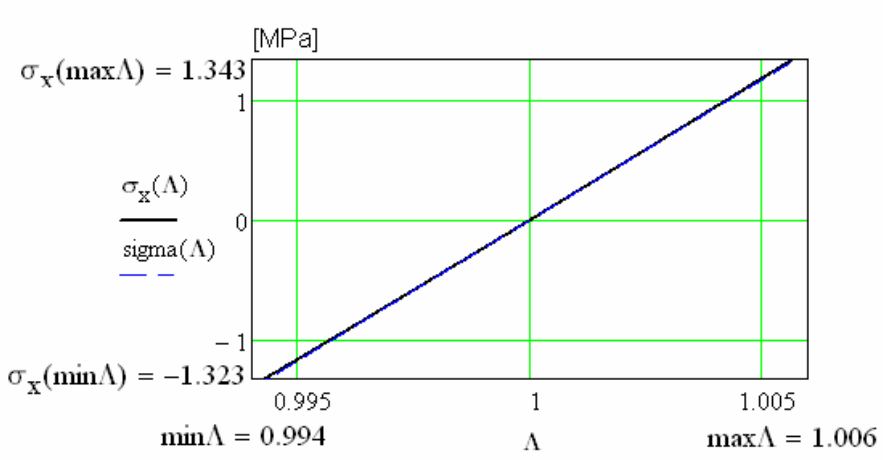
a)



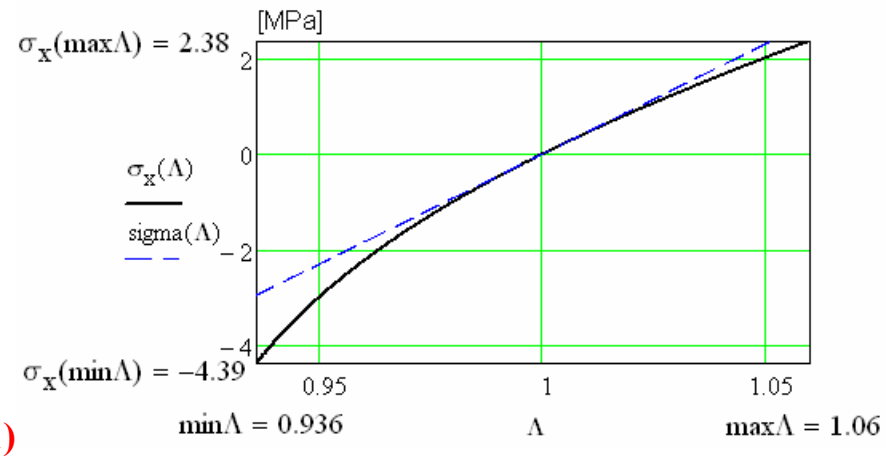
a)45



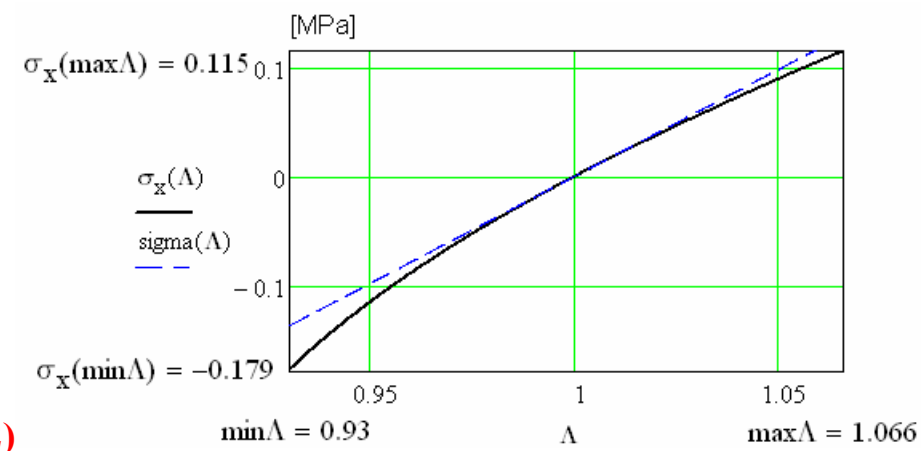
b)



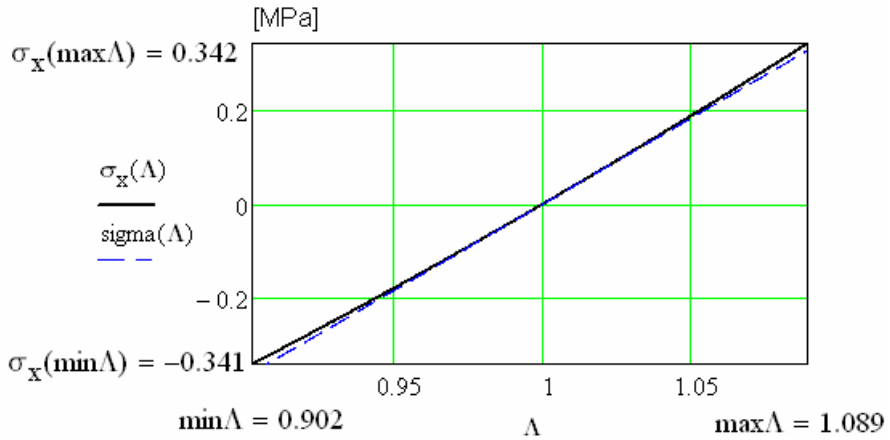
c)


[MPa]
σx(maxΛ) = 2.38
σx(minΛ) = −4.39
σx(Λ)
sigma(Λ)
2
0
−2
−4
0.95
1
1.05
minΛ = 0.936
Λ
maxΛ = 1.06

d1)

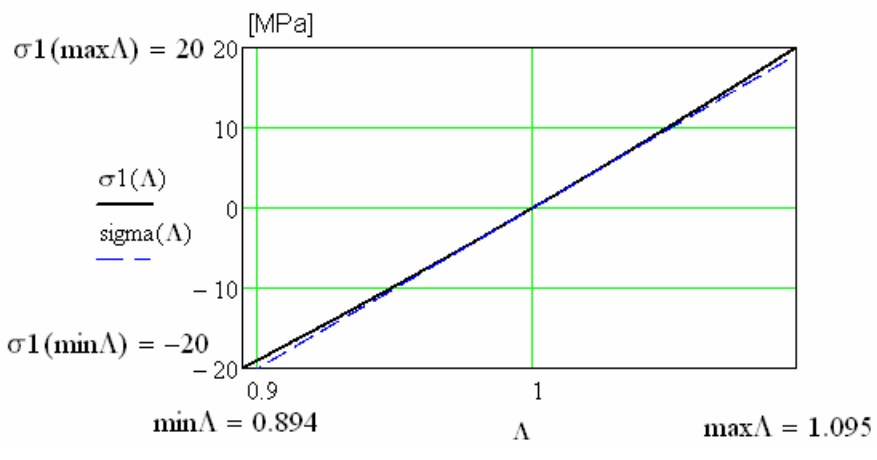

[MPa]
σx(maxΛ) = 0.115
σx(minΛ) = −0.179
σx(Λ)
sigma(Λ)
0.1
0
−0.1
0.95
1
1.05
minΛ = 0.93
Λ
maxΛ = 1.066

d2)

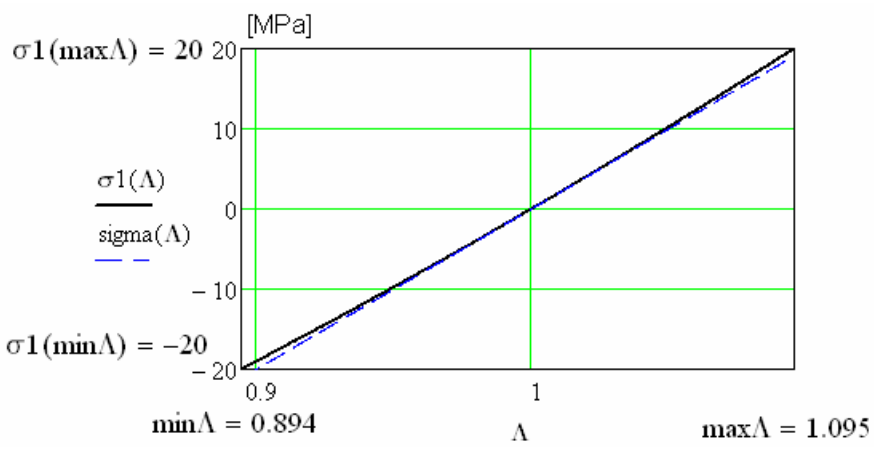

[MPa]
σx(maxΛ) = 0.342
σx(minΛ) = −0.341
σx(Λ)
sigma(Λ)
0.2
0
−0.2
0.95
1
1.05
minΛ = 0.902
Λ
maxΛ = 1.089

d3)

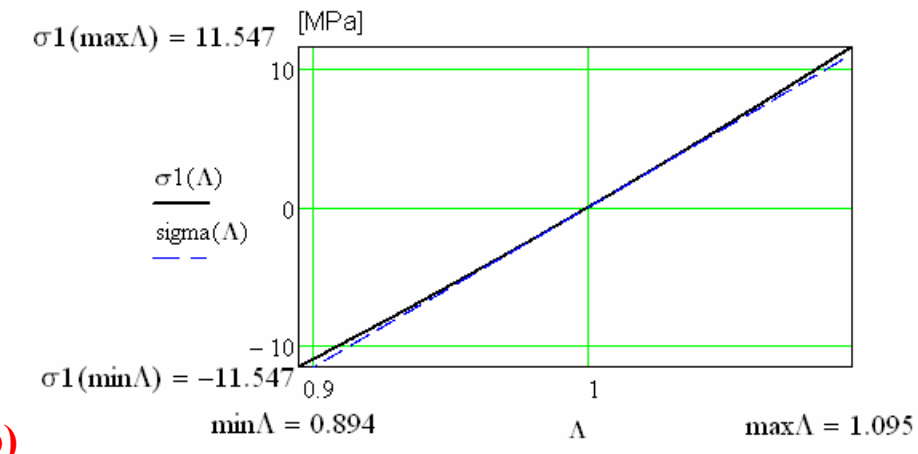
B) Biaxial tension-compression test



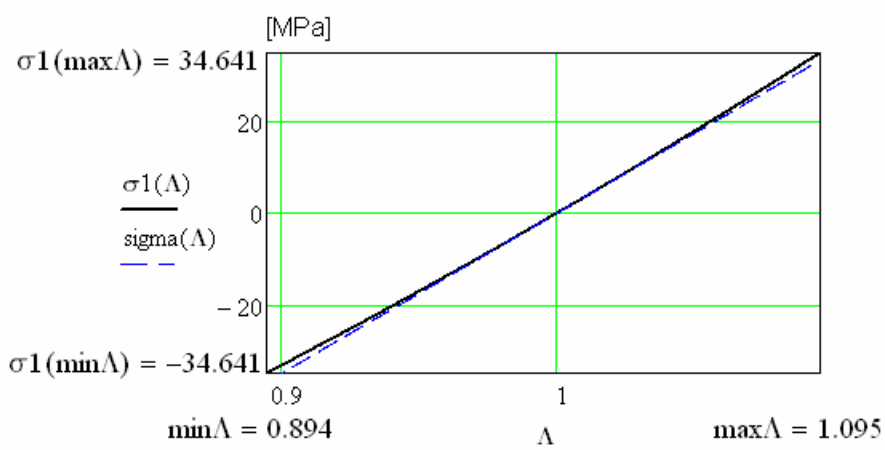
a)



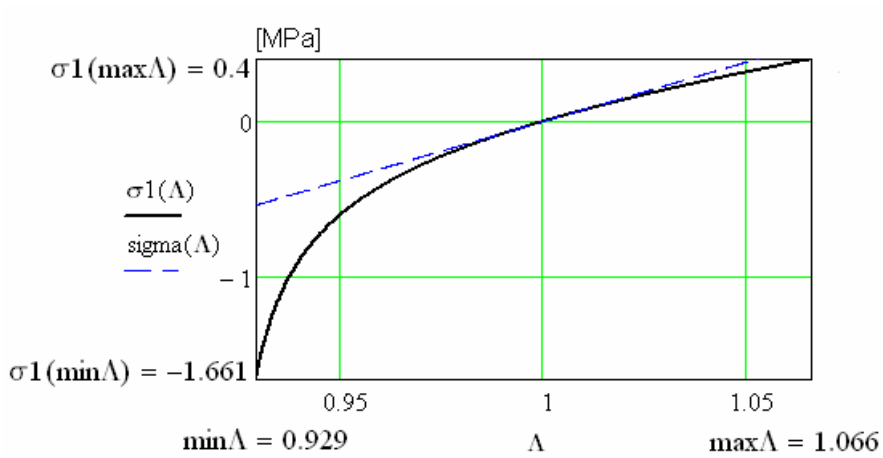
a)45



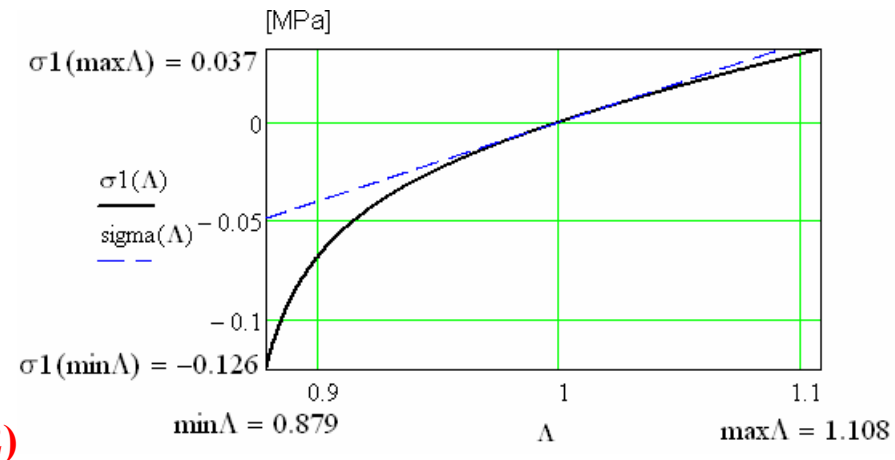
b)



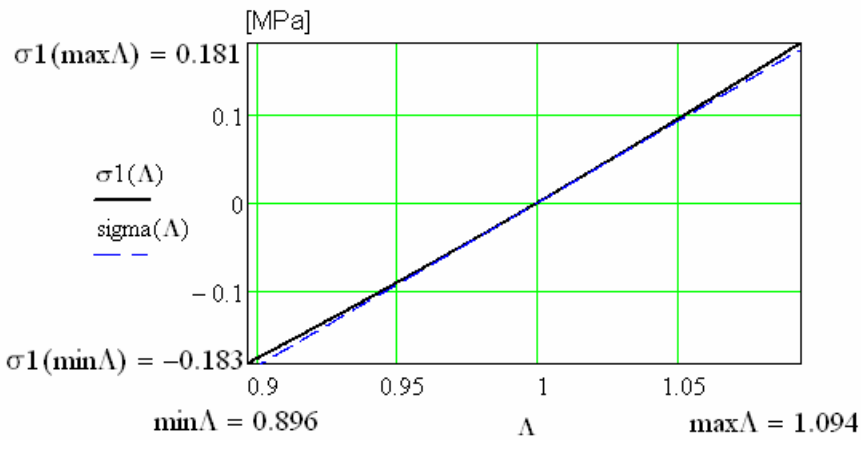
c)


[MPa]
σ1(maxΛ) = 0.4
0
σ1(Λ)
sigma(Λ)
− 1
σ1(minΛ) = −1.661
0.95
1
1.05
minΛ = 0.929
Λ
maxΛ = 1.066

d1)


[MPa]
σ1(maxΛ) = 0.037
0
σ1(Λ)
sigma(Λ)
− 0.05
− 0.1
σ1(minΛ) = −0.126
0.9
1
1.1
minΛ = 0.879
Λ
maxΛ = 1.108

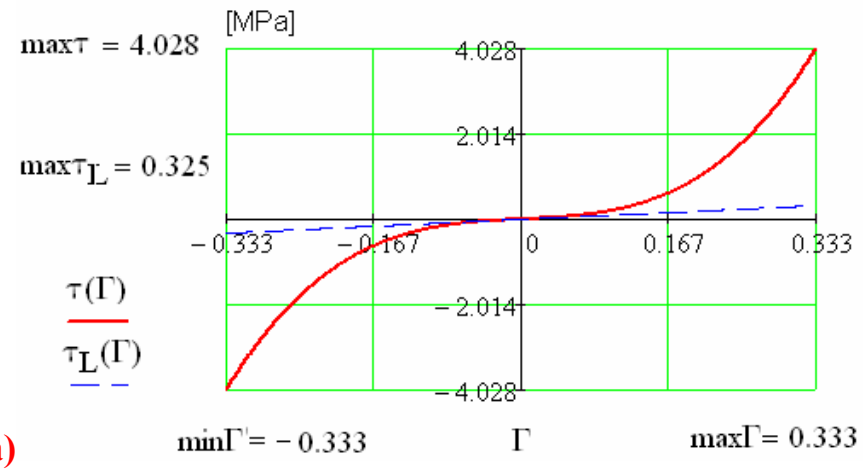
d2)

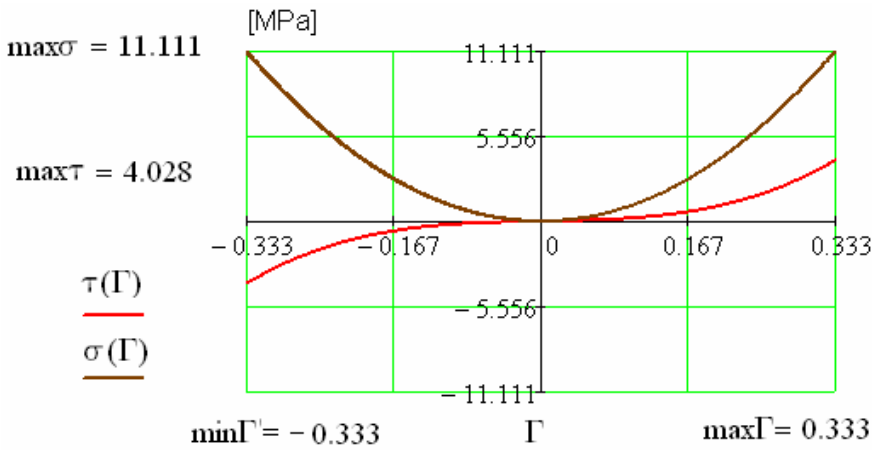

[MPa]
σ1(maxΛ) = 0.181
0.1
σ1(Λ)
sigma(Λ)
0
− 0.1
σ1(minΛ) = −0.183
0.9
0.95
1
1.05
minΛ = 0.896
Λ
maxΛ = 1.094

d3)

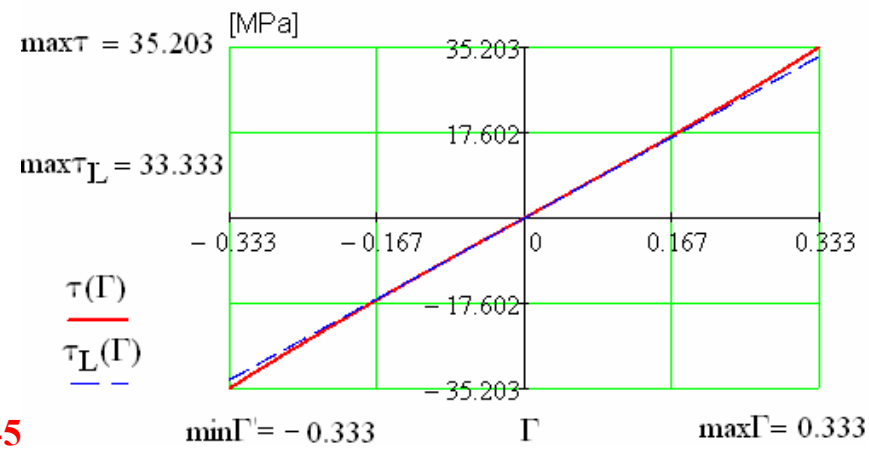
C) Simple shear deformation

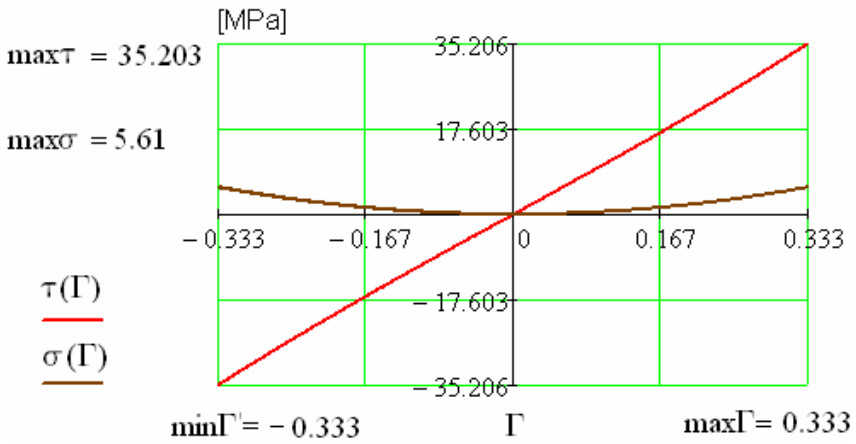
a)



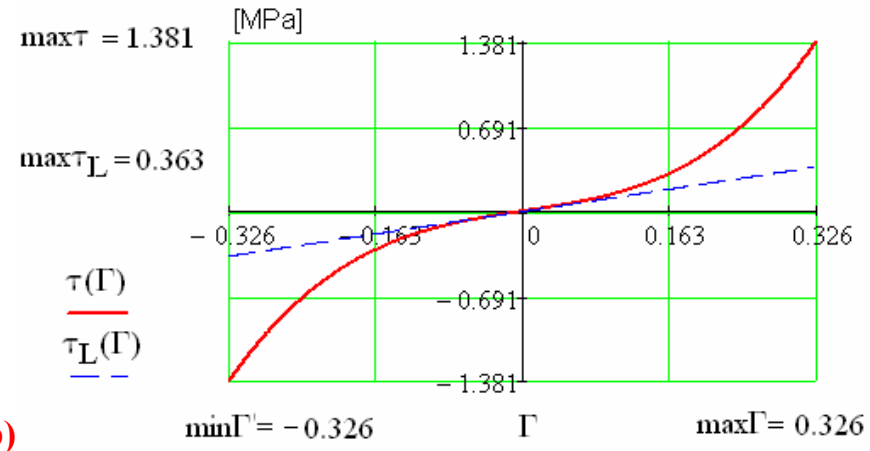


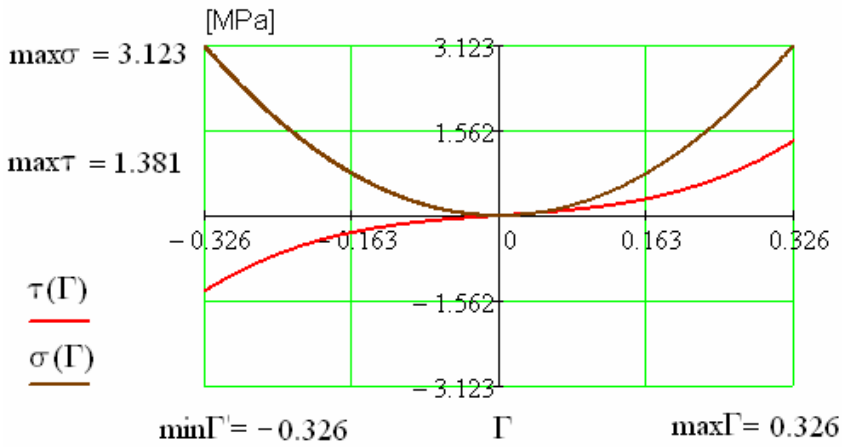
a)45



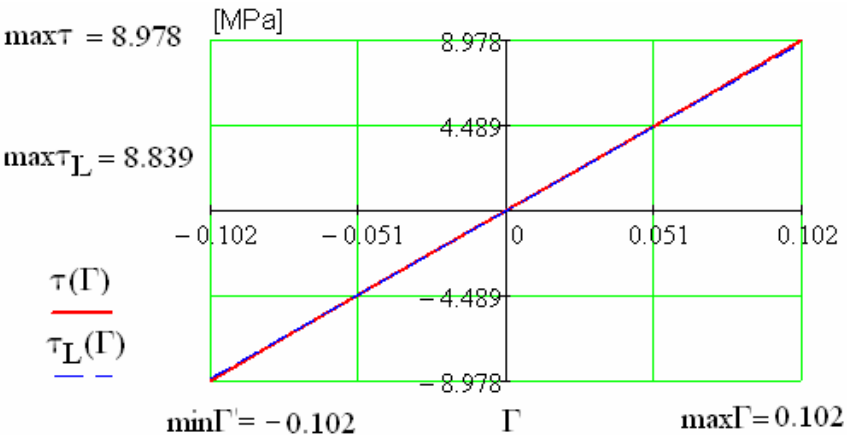


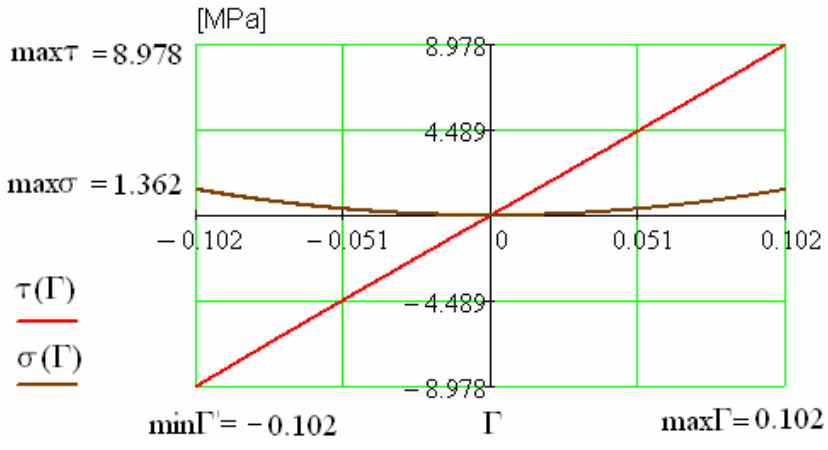
b)


maxτ = 1.381
[MPa]
maxτL = 0.363
1.381
0.691
−0.326
−0.163
0
0.163
0.326
τ(Γ)
τL(Γ)
−0.691
−1.381
minΓ = −0.326
Γ
maxΓ = 0.326

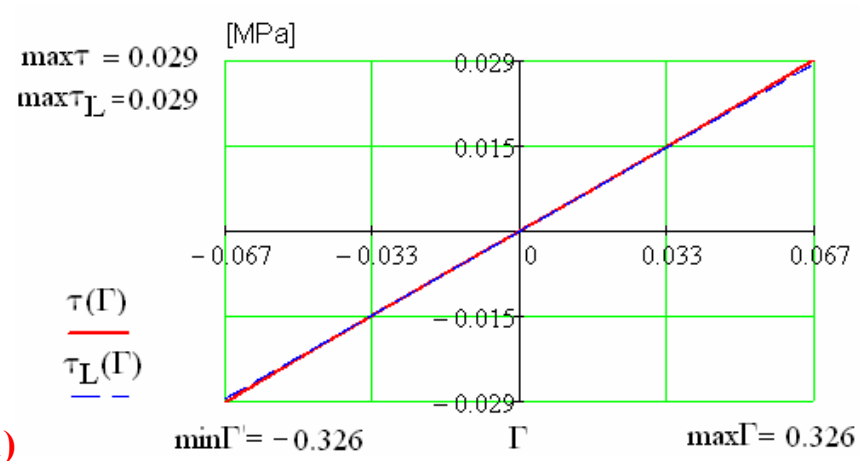

[MPa]
maxσ = 3.123
maxτ = 1.381
3.123
1.562
−0.326
−0.163
0
0.163
0.326
τ(Γ)
σ(Γ)
−1.562
−3.123
minΓ = −0.326
Γ
maxΓ = 0.326

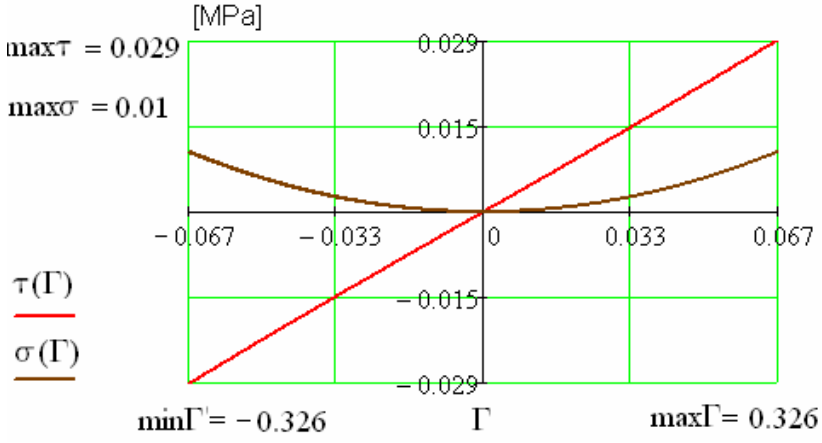
c)


maxτ = 8.978
[MPa]
maxτL = 8.839
8.978
4.489
−0.102
−0.051
0
0.051
0.102
τ(Γ)
τL(Γ)
−4.489
−8.978
minΓ = −0.102
Γ
maxΓ = 0.102

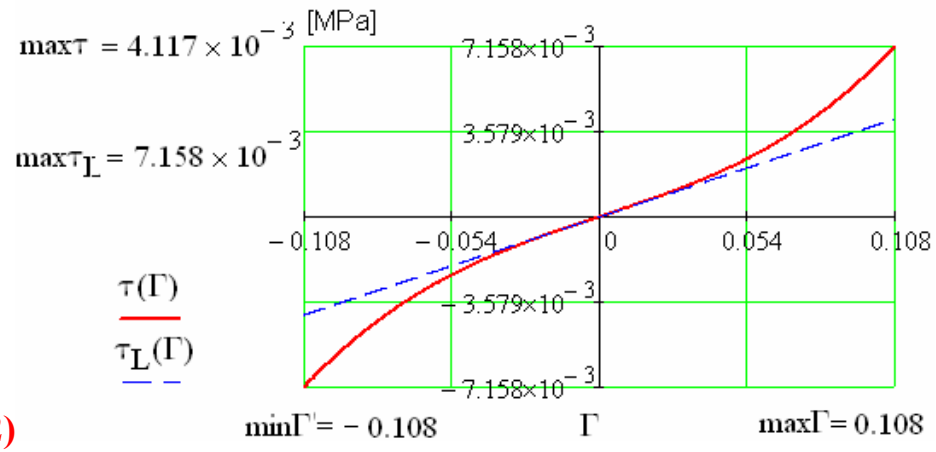

[MPa]
maxτ = 8.978
maxσ = 1.362
8.978
4.489
−0.102
−0.051
0
0.051
0.102
τ(Γ)
σ(Γ)
−4.489
−8.978
minΓ = −0.102
Γ
maxΓ = 0.102

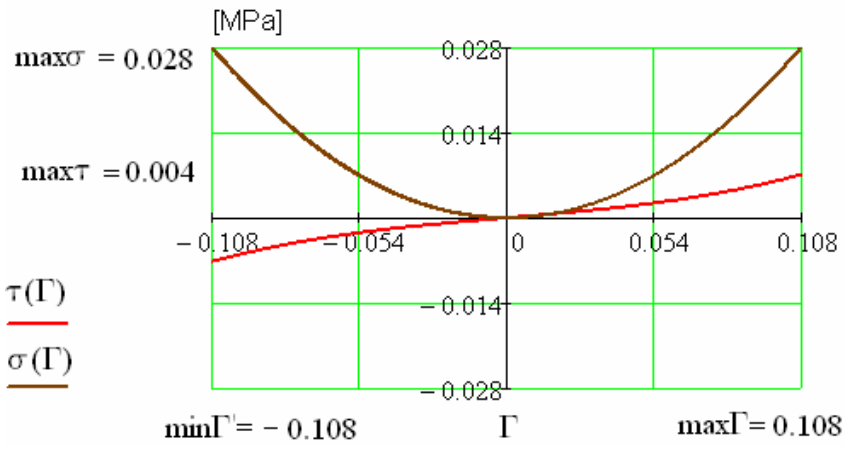
d1)


[MPa]
maxτ = 0.029
maxτL = 0.029
0.029
0.015
−0.015
−0.029
−0.067
−0.033
0
0.033
0.067
τ(Γ)
τL(Γ)
minΓ = −0.326
Γ
maxΓ = 0.326

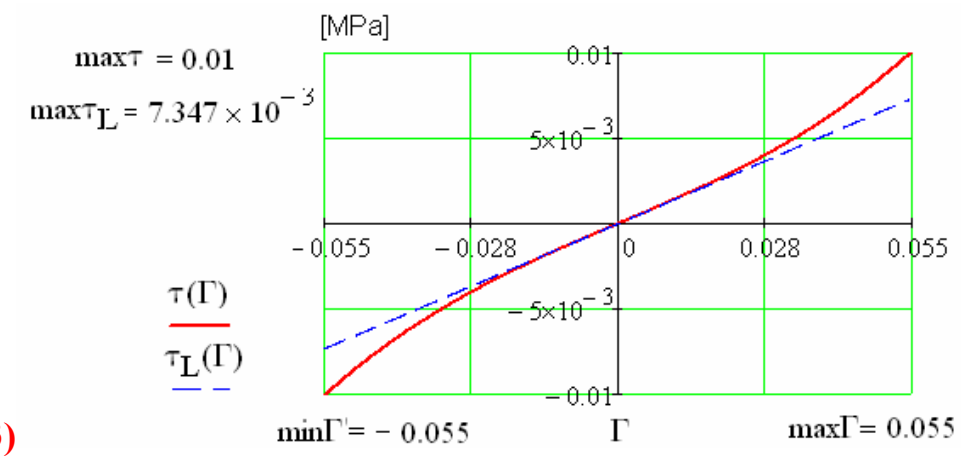

[MPa]
maxτ = 0.029
maxσ = 0.01
0.029
0.015
−0.015
−0.029
−0.067
−0.033
0
0.033
0.067
τ(Γ)
σ(Γ)
minΓ = −0.326
Γ
maxΓ = 0.326

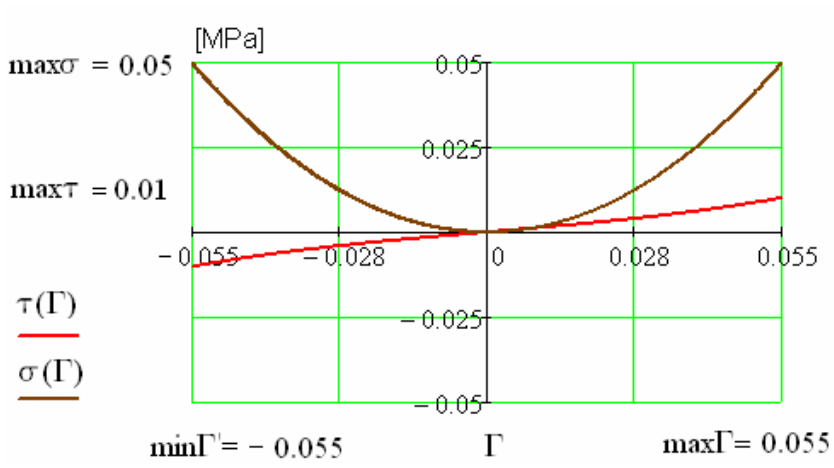
d2)


[MPa]
maxτ = 4.117 × 10−3
maxτL = 7.158 × 10−3
7.158×10−3
3.579×10−3
−3.579×10−3
−7.158×10−3
−0.108
−0.054
0
0.054
0.108
τ(Γ)
τL(Γ)
minΓ = −0.108
Γ
maxΓ = 0.108


[MPa]
maxσ = 0.028
maxτ = 0.004
0.028
0.014
−0.014
−0.028
−0.108
−0.054
0
0.054
0.108
τ(Γ)
σ(Γ)
minΓ = −0.108
Γ
maxΓ = 0.108

d3)


[MPa]
maxτ = 0.01
maxτL = 7.347 × 10−3
0.01
5×10−3
−5×10−3
−0.01
−0.055
−0.028
0
0.028
0.055
τ(Γ)
τL(Γ)
minΓ = −0.055
Γ
maxΓ = 0.055


[MPa]
maxσ = 0.05
maxτ = 0.01
0.05
0.025
−0.025
−0.05
−0.055
−0.028
0
0.028
0.055
τ(Γ)
σ(Γ)
minΓ = −0.055
Γ
maxΓ = 0.055

Conclusions

- cellulars reveal nonlinear behaviour in elastic range
- structural topology, structural elements stiffnesses influence nonlinear path
- type of material by proportion Re/Es determines the elatic range
- effect of nonlinearity is dependent on type of loading
- the model based on micromechanics requires experimental verification before application

Literature

[1] R.W. Ogden, *Nonlinear Elasticity with Application to Material Modeling*, Eds. Centre of Excellence for Advanced Materials and Structures, Warsaw (2003).

[2] V.V.Novozhilov, *Foundations of the Nonlinear Theory of Elasticity*, Dover (1999), (translated from 1948 Russian first edition).

[3] J.Bonet, A.J.Burton, A simple orthotropic, transversely isotropic hyperelastic constitutive equation for large strain computations, Comp. Meth. in Appl. Mech. and Engng. 162,151-164, (1998).

[4] J.E.Bischoff, E.M.Arruda, K.Grosh, Finite element simulation of orthotropic hyper-elasticity, Finite Element Analysis and Design, 38,983-998, (2002).

[5] P. Dłużewski, P.Rodzik, Elastic eigenstates in finite element modeling of large anisotropic elasticity, Comp.Meth. Appl. Mech. Engng. 160, 325-335, (1998).

[6] J. Hohe, W.Becker, Effective mechanical behaviour of hyperelastic honeycombs and twodimensional model foams at finite strain, Int. Journ. of Mech. Sci., 45, 891-913, (2003).

[7] S.Nemat-Naser, M.Hori, *Micromechanics,.* 2nd edition Elsevier 1999.

[8] W.E.Warren, A.M.Kraynik, The Nonlinear Elastic Behaviour of Open-Cell Foams, Transactions of ASME, 58, 375-381, June (1991).

[9] W.E. Warren, A.M. Kraynik, C.M. Stone, A Constitutive Model for Two-Dimensional Nonlinear Elastic Foams, J.Mech. Phys. Solids and Struct. 37, 717-733, (1989).

[10] M. Janus-Michalska , Effective Models Describing Elastic Behaviour of Cellular Materials, Arch. Met. and Mat.,50,595-608, (2005).

[11] M.M. Attard, Finite strain isotropic hyperelasticity, Int. Jour. of Solids and Struct., 40, 4353-4378, (2003).

[12] A. Menzel, P.Steimann, A view on anisotropic finite hyperelasticity, European Journal of Mechanics A/Solids, 22, 71-87, (2003).

[13] A.J.Wang, D.L. McDowell, In-Plane Stiffness and Yield Strength of Periodic Metal Honeycombs., Journal of Engineering Materials and Technology, Vol 126, pp.137-156.