ENERGY-BASED APPROACH TO LIMIT STATE CRITERIA OF CELLULAR MATERIALS

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ABSTRACT

Formulation of limit state criteria derived on micromechanical analysis for open-cell anisotropic media modeled as periodic beam structure is presented. Linear response and material strength from macroscopic perspective is described by energy based constitutive model.

PROBLEM DEFINITION

- multiscale modeling
- formulation for equivalent continuum

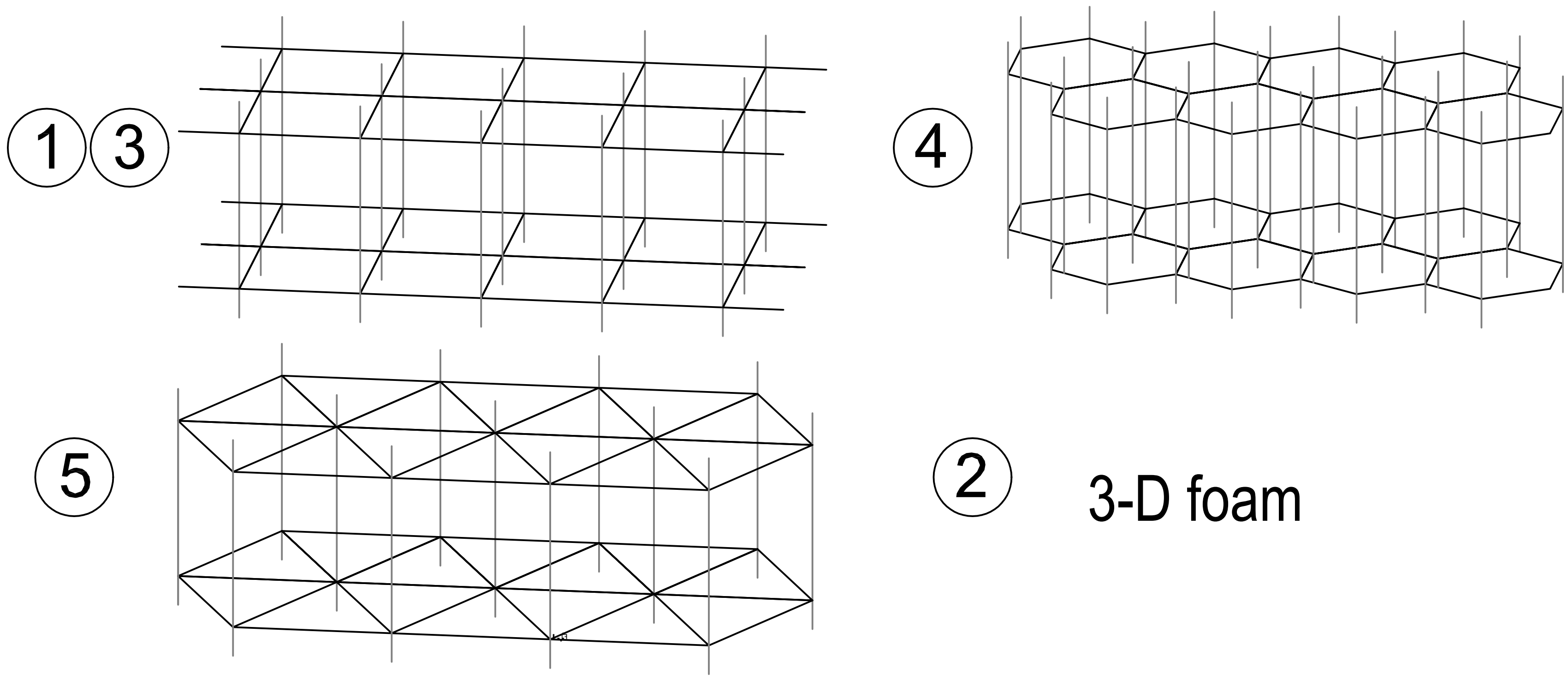
  Rychlewski criterion

$$\frac{\Phi(\boldsymbol{\sigma}_I)}{\Phi_I^{cr}} + \frac{\Phi(\boldsymbol{\sigma}_{II})}{\Phi_{II}^{cr}} + \ldots\ldots \frac{\Phi(\boldsymbol{\sigma}_\rho)}{\Phi_\rho^{cr}} \leq 1 \qquad \rho \leq 6$$

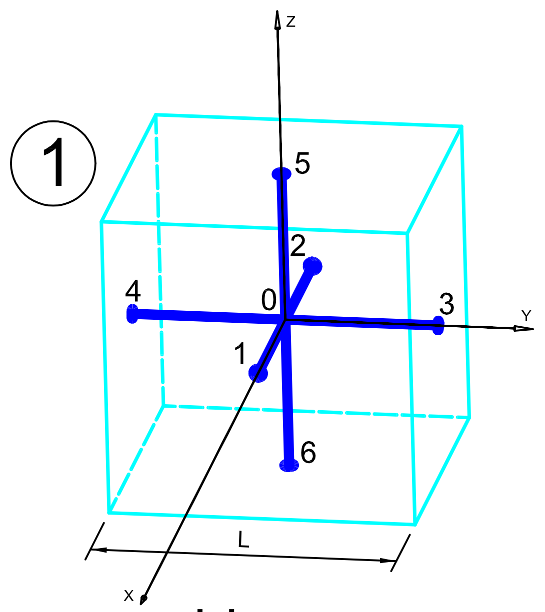
$\boldsymbol{\sigma}_I \ldots, \boldsymbol{\sigma}_\rho$ eigenstates

$\Phi_I^{cr} \ldots, \Phi_\rho^{cr}$ critical energy densities

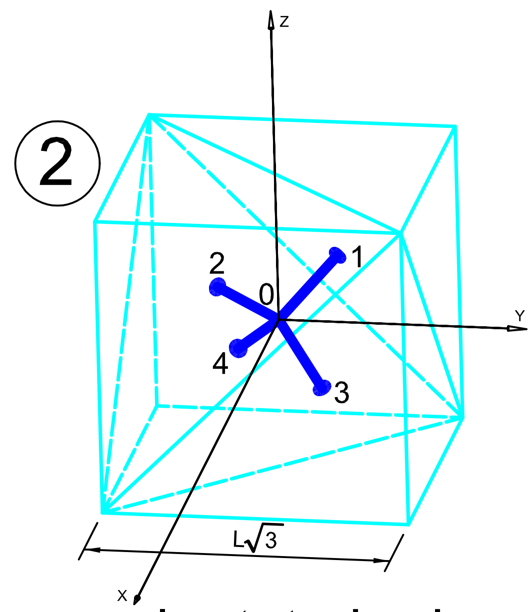
Microstructures



Representative unit cells

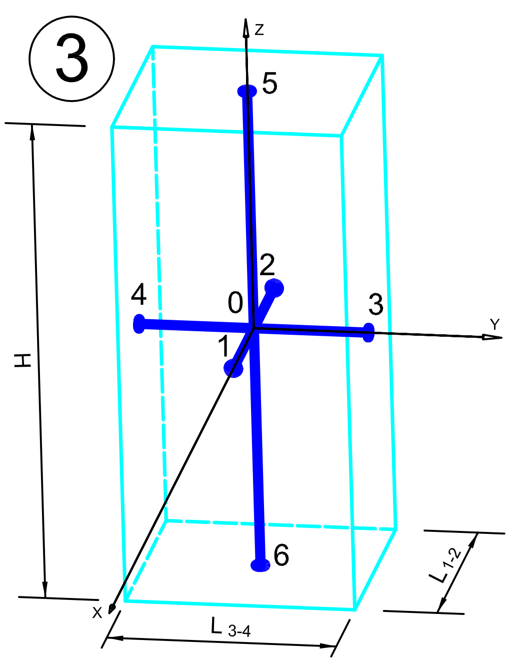


cubic

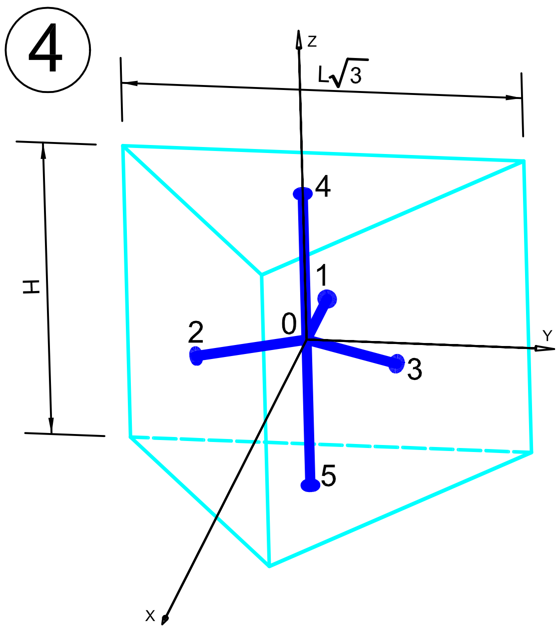


regular tetrahedron

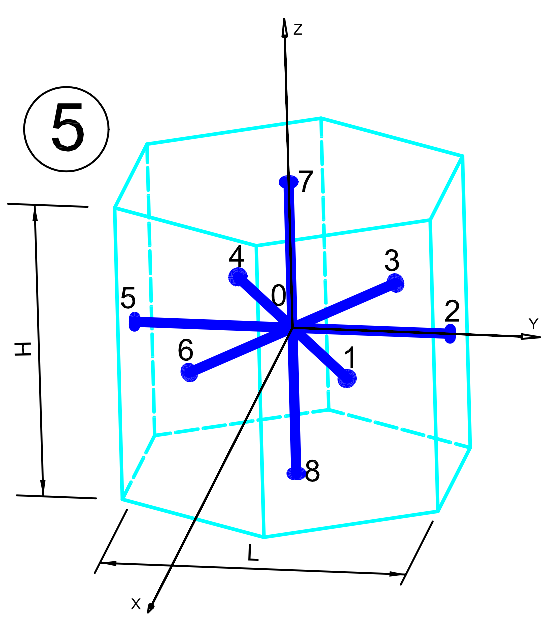
cubic symmetry



rectangular prism



trigonal prism



hexagonal prism

ortothropic symmetry

transversaly isotropic symmetry

METHOD of structural mechanics

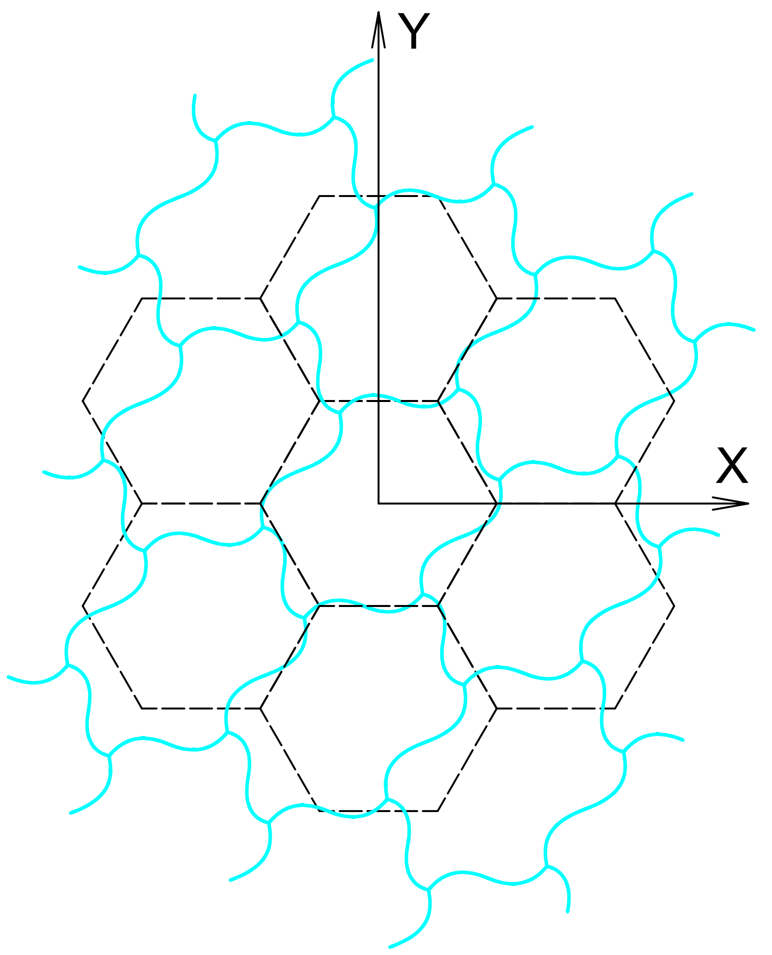
Framework of micromechanical analysis

- representative unit cell - kinematics
- mechanical model Timoshenko beam
- stress tensor definition
- stiffness tensor: Kelvin moduli and eigenstates
- micro-macro transition

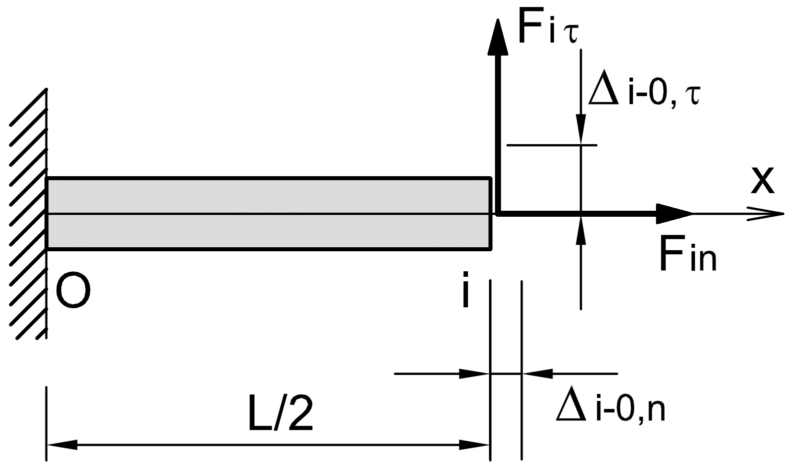
Kinematics

- affinity of nodes displacements
- uniform states of strains (macro-scale)

$$\mathbf{\Delta}_i = \mathbf{\Delta}_{i-0} + \boldsymbol{\psi} \times \mathbf{b}_i^0 + \mathbf{\Delta}_0$$



Timoshenko beam model



axial strut compliance

$$c_n = s_n^{-1} = \frac{L}{2E_S A}$$

bending strut compliance

$$c_\tau = s_\tau^{-1} = \frac{L^3}{24E_S J} + \frac{L}{2G_S A_\tau}$$

s skeleton material

Displacement-force relations

$$F_{in} = \Delta_{i-0,n} \cdot s_n$$

$$F_{i\tau} = \Delta_{i-0,\tau} \cdot s_\tau$$

Stress definition for equivalent continuum

$$\boldsymbol{\sigma} = \frac{1}{V} \int_{V^s} \boldsymbol{\sigma}^s dV$$

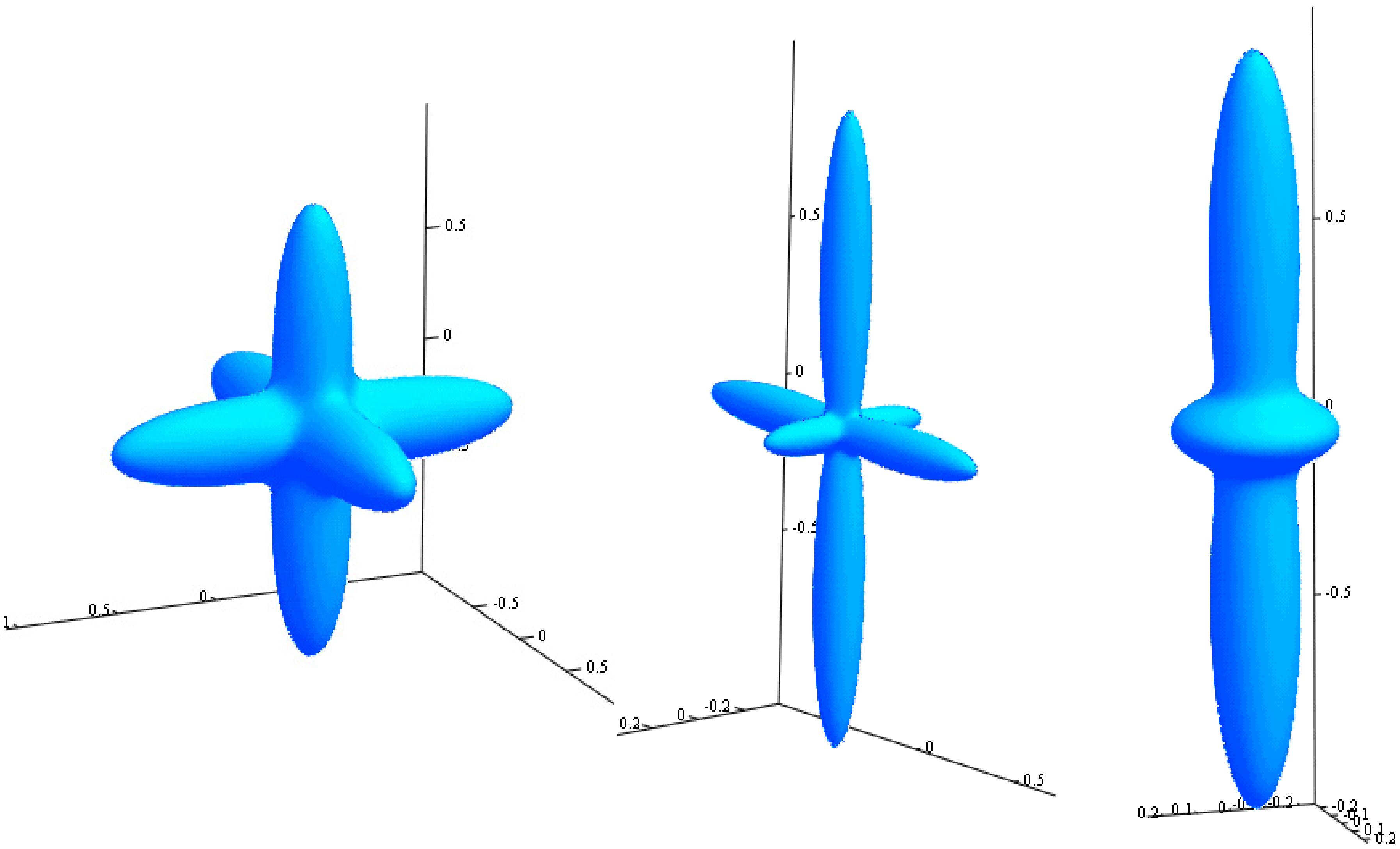
Hooke's law $\boldsymbol{\sigma} = \mathbf{S}\ \boldsymbol{\varepsilon}$,

$\mathbf{S}$ stiffness tensor, λ_i eigenvalues

$\boldsymbol{\sigma}_i$ eigenstates $i = 1,\ldots\rho,\ \rho \le 6$

Young's moduli $E(\mathbf{n}) = (\mathbf{n} \otimes \mathbf{n}) \cdot \mathbf{S} \cdot (\mathbf{n} \otimes \mathbf{n})$

$$E_r(\mathbf{n}) = E(\mathbf{n}) / E_{\max}$$



cubic　　rectangular prism　　trigonal prism

LIMIT STATES

skeleton

the limit states in the skeleton are calculated with the use of Huber criterion

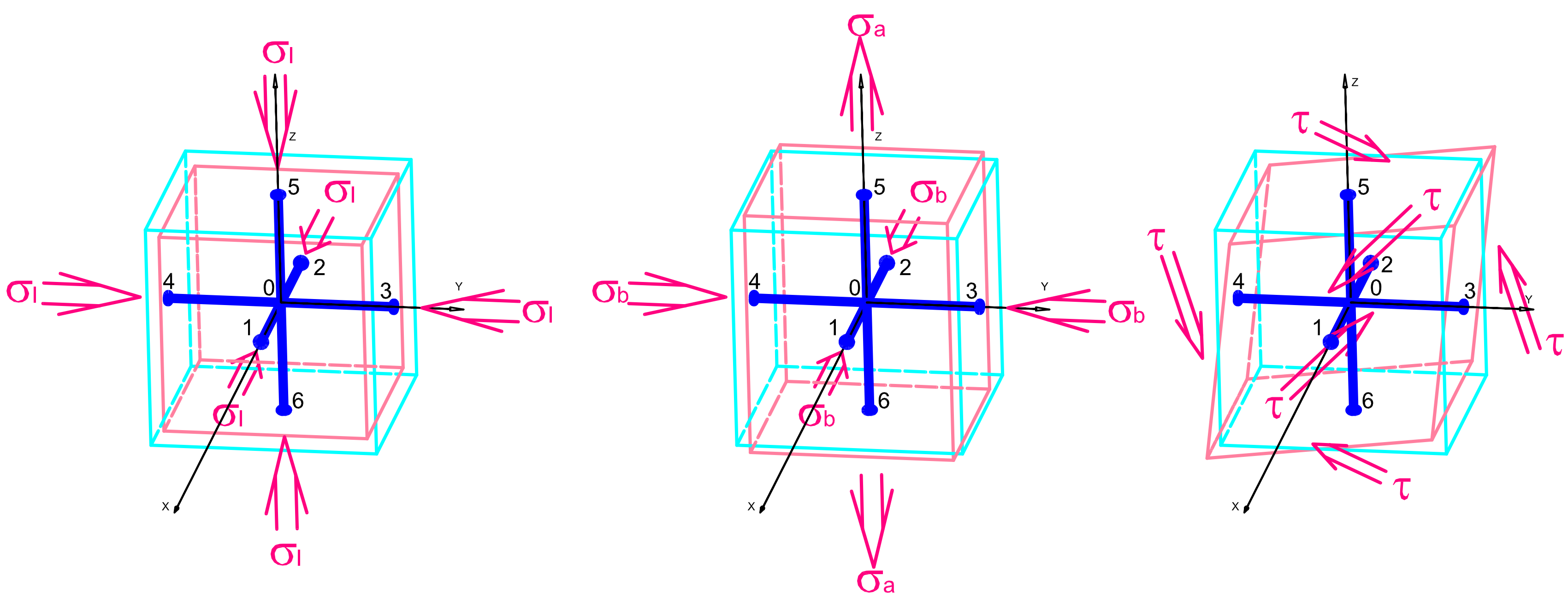
$\Leftrightarrow$

cellular material

(as a continuum) limit elastic eigenstates

$$\boldsymbol{\sigma}^{s}_{i,cr} \Leftrightarrow \boldsymbol{\sigma}_{i,cr}$$

$\Downarrow$ critical energy densities $\phi_{i,cr}$



$$\boldsymbol{\sigma}_{\mathbf{I}} = \begin{bmatrix} \sigma_I & 0 & 0 \\ 0 & \sigma_I & 0 \\ 0 & 0 & \sigma_I \end{bmatrix} \quad \boldsymbol{\sigma}_{\mathbf{II}} = \begin{bmatrix} \sigma_a & 0 & 0 \\ 0 & \sigma_b & 0 \\ 0 & 0 & \sigma_b \end{bmatrix} \quad \boldsymbol{\sigma}_{\mathbf{III}} = \begin{bmatrix} 0 & \tau & \tau \\ \tau & 0 & \tau \\ \tau & \tau & 0 \end{bmatrix}$$

$$\sigma_s = R_e \quad \Rightarrow \quad \boldsymbol{\sigma}_i = \boldsymbol{\sigma}_{i,cr}$$

$$\phi_{i,cr} = \frac{1}{\lambda_i} \boldsymbol{\sigma}_{i,cr} \cdot \boldsymbol{\sigma}_{i,cr}$$

RESULTS

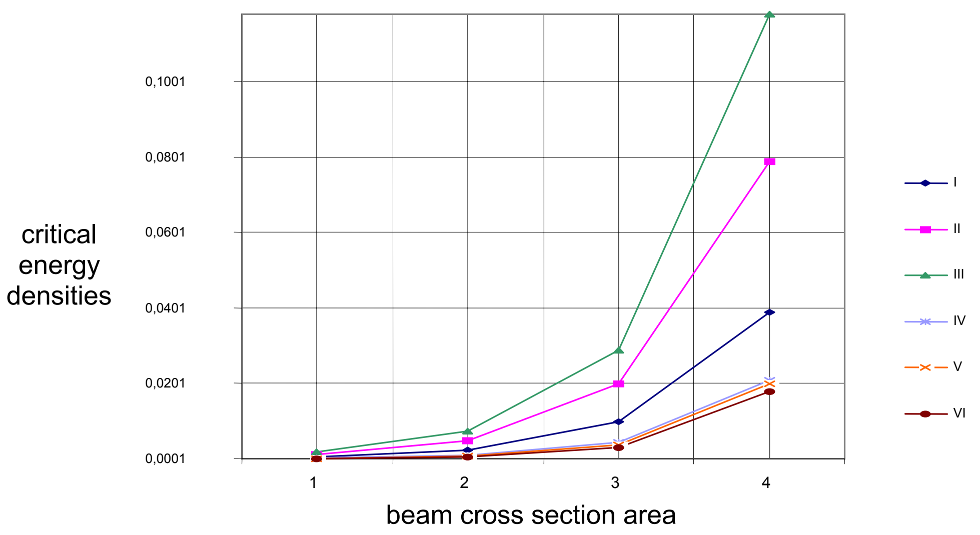
eigenvalues $\lambda_i = \lambda_i(s_n, s_\tau, H, L, A, ...)$

cubic cell	$\lambda_I = \lambda_1 = \frac{s_n}{2L}, \quad \lambda_{II} = \lambda_2 = \lambda_3 = \frac{s_n}{2L}, \quad \lambda_{III} = \lambda_4 = \lambda_5 = \lambda_6 = \frac{s_\tau}{2L}$
rectangular prism	$\lambda_I = \lambda_1 = \frac{L_{1-2}\, s_{n1-2}}{2\, L_{3-4}\, H}, \quad \lambda_{II} = \lambda_2 = \frac{L_{3-4}\, s_{n3-4}}{2\, L_{1-2}\, H}, \quad \lambda_{III} = \lambda_3 = \frac{H\, s_{n5-6}}{2\, L_{1-2}\, L_{3-4}}$ $\lambda_{IV} = \lambda_4 = \frac{\frac{2\, H^2\, s_{\tau 5-6}}{L_{3-4}{}^2\, s_{\tau 3-4} + H^2\, s_{\tau 5-6}} \; \frac{L_{3-4}}{2}\, s_{\tau 3-4}}{L_{1-2}\, H},$ $\lambda_V = \lambda_5 = \frac{\frac{2\, H^2\, s_{\tau 5-6}}{L_{1-2}{}^2\, s_{\tau 1-2} + H^2\, s_{\tau 5-6}} \; \frac{L_{1-2}}{2}\, s_{\tau 1-2}}{L_{3-4}\, H}$ $\lambda_{VI} = \lambda_6 = \frac{\frac{2\, L_{3-4}{}^2\, s_{\tau 3-4}}{L_{3-4}{}^2\, s_{\tau 3-4} + L_{1-2}{}^2\, s_{\tau 1-2}} \; \frac{L_{12}}{2}\, s_{\tau 1-2}}{L_{3-4}\, H}$
trigonal prism	$\lambda_I = \lambda_1 = \frac{2\sqrt{3}\, s_{nL}}{9\, L}, \quad \lambda_{II} = \lambda_3 = \frac{\sqrt{3}\, s_{nL}}{6\, H}, \quad \lambda_{III} = \lambda_2 = \lambda_6 = \frac{\sqrt{3}\, s_{nL}\, s_{\tau L}}{3\, H\,(s_{nL} + s_{\tau L})},$ $\lambda_{IV} = \lambda_4 = \lambda_5 = \frac{4\sqrt{3}\, H\, s_{\tau H}\, s_{\tau L}}{3\, L^2\, s_{\tau L} + 4\, H^2\, s_{\tau H}}$
foam cell	$\lambda_I = \lambda_1 = \frac{2(s_n + 2s_\tau)}{9\sqrt{3}\, L}, \quad \lambda_{II} = \lambda_2 = \lambda_3 = \frac{2(s_n - s_\tau)}{9\sqrt{3}\, L}, \quad \lambda_{III} = \lambda_{II}$

energy densities $\Phi_i = \Phi_i(R_e, s_n, s_\tau, H, L, A, ...)$

cubic cell	$\Phi_I^{gr} = \frac{1}{\lambda_I}\, 3\left(\frac{A\, R_e}{L^2}\right)^2, \; \Phi_{II}^{gr} = \frac{1}{\lambda_I}\, \frac{2}{3}\left(\frac{A\, R_e}{L^2}\right)^2, \; \Phi_{III}^{gr} = \frac{1}{\lambda_{III}}\, 24\, \frac{I^2\, R_e^2}{h^2\, L^6}$
ortotropic cell	$\Phi_I^{gr} = \frac{1}{\lambda_I}\left(\frac{A\, R_e}{L_{3-4}\, H}\right)^2, \; \Phi_{II}^{gr} = \frac{1}{\lambda_{II}}\left(\frac{A\, R_e}{L_{3-4}\, H}\right)^2, \; \Phi_{III}^{gr} = \frac{1}{\lambda_{III}}\left(\frac{A\, R_e}{L_{1-2}\, L_{3-4}}\right)^2$ $\Phi_{IV}^{gr} = \frac{1}{\lambda_{IV}}\, 8\, \frac{I^2\, R_e^2}{h^2\, L_{1-2}{}^2\, L_{3-4}{}^2\, H^2}, \quad \Phi_V^{gr} = \frac{1}{\lambda_V}\, 8\, \frac{I^2\, R_e^2}{h^2\, L_{1-2}{}^2\, L_{3-4}{}^2\, H^2}$ $\Phi_{VI}^{gr} = \frac{1}{\lambda_{VI}}\, 8\, \frac{I^2\, R_e^2}{h^2\, L_{1-2}{}^2\, L_{3-4}{}^2\, H^2}.$
trigonal prism	$\Phi_I^{gr} = \frac{1}{\lambda_I}\, \frac{(A\, R_e)^2}{2L\, H^3}, \; \Phi_{II}^{gr} = \frac{1}{\lambda_{II}}\, \frac{64\; H^2\; A^2\; R_e^2}{3\, L^4}$ $\Phi_{III}^{gr} = \frac{26\sqrt{3}\, s_{\tau L}\,(s_{nL} + s_{\tau L})\, I^2\; A^2\; R_e{}^2}{9\, s_{nL}\; H\, L^2\,(2\, I\, s_{nL} + 2\, I\, s_{\tau L} + L\, s_{\tau L}\; h\; A)^2},$ $\Phi_{IV}^{gr} = \frac{16\sqrt{3}\,(3\, L^2\, s_{\tau L} + 4\, H^2\, s_{\tau H})\, I^2\; R_e^2}{27\, H^3\, s_{\tau H}\; s_{\tau L}\; L^4\; h^2}$
foam cell	$\Phi_I^{gr} = \frac{2}{3K}\left(\frac{\rho}{\rho^s}\right)^2 R_e^2, \; \Phi_{II}^{gr} = \frac{R_e^2}{4G}\, f(A, L)$

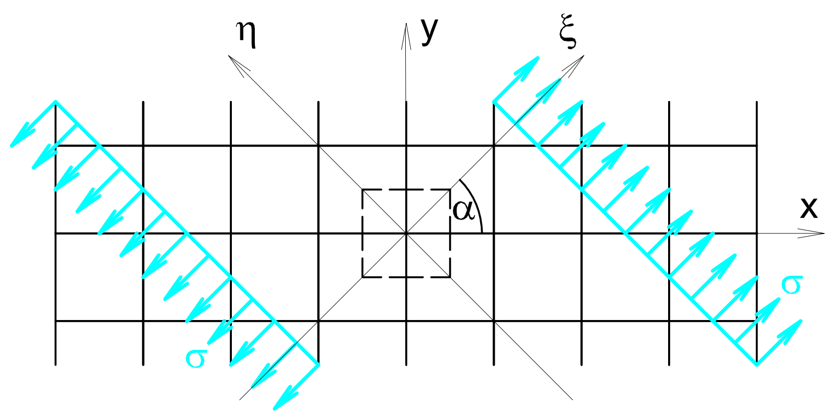
DISTRIBUTION OF ENERGY LIMITS



- dependence on microstructure parameters
- modelling possibilities

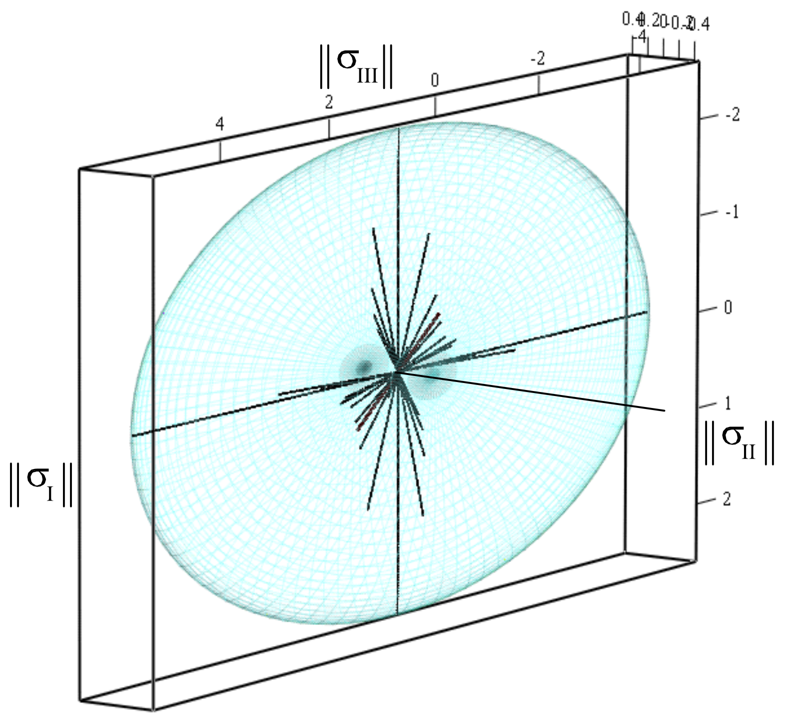
EXPERIMENTAL VERIFICATION OF ENERGY CRITERION

tension-compression tests in xy plane



$$\boldsymbol{\sigma}(\xi,\eta)=\begin{bmatrix}\sigma & 0\\ 0 & 0\end{bmatrix}$$

α - cell orientation angle with respect to tension direction



$$\boldsymbol{\sigma}=\boldsymbol{\sigma}_{\mathbf{I}}+\boldsymbol{\sigma}_{\mathbf{II}}+\boldsymbol{\sigma}_{\mathbf{III}}$$

eigenstate decomposition of the plane stress state

theoretical prediction of σ_{cr}

$$\sigma_{cr}=\left[\frac{1}{3\lambda_I\Phi_{I,cr}}+\frac{4-3(\sin 2\alpha)^2}{6\lambda_{II}\Phi_{II,cr}}+\frac{(\sin 2\alpha)^2}{2\lambda_{III}\Phi_{III,cr}}\right]^{-1/2}$$ for cubic cell

CONCLUSION

- effective model of elastic behaviour and limit state of open-cell microstructures is proposed
- such an approach can be used for each type of (micro)structure (topology and morphology)
- the presented analysis can be extended for:

  nonlinear elasticity
  plastic analysis of struts (plastic hinges)
  failure analysis
  different models (plate model)

- theoretical background for experiment is given

Literature

[1] **L.J. Gibson, M.F. Ashby** (1997). ***Cellular Solids,*** 2nd edition Cambridge University Press.

[2] **J.Rychlewski** (1984).Unconventional approach to linear elasticity, *Arch. Mech.,* **47**, 1995, 149-171.

[3] **J.Ostrowska-Maciejewska, J.Rychlewski,** (1988). Plane elastic and limit states in anisotropic solids, Arch. Mech., **40**, 379-386.

[4] **M.Janus-Michalska, R.B.Pęcherski, (**2003). Macroscopic properties of open-cell foams based on micromechanical modelling, Technische Mechanik.

[5] **P.Kordzikowski, M.Janus-Michalska, R.B.Pęcherski, (**2003). Analysis of the influence of the strength of the struts forming a cubic cell structure on the distribution of the energy limits, Rudy i Metale Nieżelazne, R49,No.3,2004.