W punkcie ciała dana jest macierz naprężeń: $\mathbf{T}_\sigma$. Znaleźć w tym punkcie naprężenia główne i ich kierunki.

$$\mathbf{T}_\sigma = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 2 \\ 2 & 2 & 0 \end{bmatrix} \text{MPa}$$

Rozwiązanie:

Naprężenia główne = wartości własne tensora $\mathbf{T}_\sigma$

$$\det (\mathbf{T}_\sigma - \mathbf{I}\, \sigma_{(i)}) = 0 \qquad \begin{vmatrix} 2-\sigma & 0 & 2 \\ 0 & 1-\sigma & 2 \\ 2 & 2 & -\sigma \end{vmatrix} = 0$$

$$\sigma_{(i)}^3 - I_1\, \sigma_{(i)}^2 + I_2\, \sigma_{(i)} - I_3 = 0$$

$$I_1 = 3\,, \quad I_2 = \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 2 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} = -6\,, \quad I_3 = \begin{vmatrix} 2 & 0 & 2 \\ 0 & 1 & 2 \\ 2 & 2 & 0 \end{vmatrix} = -12 \quad \{\text{MPa}\}$$

$$\sigma_{(i)}^3 - 3\,\sigma_{(i)}^2 - 6\,\sigma_{(i)} + 12 = 0$$

$$a\,\sigma_{(i)}^3 + b\,\sigma_{(i)}^2 + c\,\sigma_{(i)} + d = 0$$

Wzory Cardano

$\sigma_{(i)} = x - b/(3*a) = x + 1$ $\qquad$ $a=1$

$p = c - b^2/3 = -6 -3 = -9$

$q = 2* b^3/27 - b*c/3 + d = -2 -6 +12 = 4$

$x^3 + p\, x + q = 0$, $\qquad x^3 - 9\, x + 4 = 0$

$$r = \sqrt{\frac{-p}{3}} = \sqrt{3} = 1.73205$$

$$\phi = \arccos\left(\frac{-q}{2r^3}\right) = \arccos\left(\frac{-4}{6\sqrt{3}}\right) = 1.9659 \text{ rad}$$

$$x_1 = 2\, r \cos\left(\frac{\phi}{3}\right) = 2.7466 \qquad \sigma_{(1)} = 3.7466 \text{ MPa}$$

$$x_3 = 2\, r \cos\left(\frac{\phi}{3} + \frac{2\pi}{3}\right) = -3.2015 \qquad \sigma_{(3)} = -2.2015 \text{ MPa}$$

$$x_2 = 2\, r \cos\left(\frac{\phi}{3} + \frac{4\pi}{3}\right) = 0.4549 \qquad \sigma_{(2)} = 1.4549 \text{ MPa}$$

Kierunki główne = wektory(wersory) własne tensora $\mathbf{T}_\sigma$

$(\mathbf{T}_\sigma - \mathbf{I}\,\sigma_{(i)})\,\mathbf{v}_i = \mathbf{0}$

dla $\sigma_{(3)} = -2.2015$ MPa : $\mathbf{v}_3 = (\alpha_{31}, \alpha_{32}, \alpha_{33})^T$

$(2-(-2.2015))\alpha_{31} + 0\,\alpha_{32} + 2\,\alpha_{33} = 0$ (1.1)

$0\,\alpha_{31} + (1-(-2.2015))\alpha_{32} + 2\,\alpha_{33} = 0$ (1.2)

$2\,\alpha_{31} + 2\,\alpha_{32} + (0-(-2.2015))\alpha_{33} = 0$ (1.3)

$(\alpha_{31}^{\ 2} + \alpha_{32}^{\ 2} + \alpha_{33}^{\ 2})^{0.5} = 1$ (1.4)

z (1.1): $4.2015\,\alpha_{31} + 2\,\alpha_{33} = 0 \Rightarrow \alpha_{31} = -0.476\,\alpha_{33} = -0.374$

z (1.2): $3.2015\,\alpha_{32} + 2\,\alpha_{33} = 0 \Rightarrow \alpha_{32} = -0.625\,\alpha_{33} = -0.4915$

do (1.4): $\alpha_{33}\,(0.476^{2} + 0.625^{2} + 1)^{0.5} = 1 \Rightarrow \alpha_{33} = 0.7864$ -^

dla $\sigma_{(2)} = 1.4549$ MPa : $\mathbf{v}_2 = (\alpha_{21}, \alpha_{22}, \alpha_{23})^T$

$(2-1.4549)\alpha_{21} + 0\,\alpha_{22} + 2\,\alpha_{23} = 0$ (2.1)

$0\,\alpha_{21} + (1-1.4549)\alpha_{22} + 2\,\alpha_{23} = 0$ (2.2)

$2\,\alpha_{21} + 2\,\alpha_{22} + (0-1.4549)\alpha_{23} = 0$ (2.3)

$(\alpha_{21}^{\ 2} + \alpha_{22}^{\ 2} + \alpha_{23}^{\ 2})^{0.5} = 1$ (2.4)

z (2.1): $0.546\,\alpha_{21} + 2\,\alpha_{23} = 0 \Rightarrow \alpha_{21} = -3.663\,\alpha_{23} = -0.63$

z (2.2): $-0.455\,\alpha_{22} + 2\,\alpha_{23} = 0 \Rightarrow \alpha_{22} = 4.405\,\alpha_{23} = 0.758$

do (2.4): $\alpha_{23} = 1/(3.663^{2} + 4.405^{2} + 1)^{0.5} = 0.172$ -^

dla $\sigma_{(1)} = 3.747$ MPa

$\mathbf{v}_1 = \mathbf{v}_2 \times \mathbf{v}_3 = (0.679, 0.432, 0.593)^T$

$\mathbf{v}_2 = \mathbf{v}_3 \times \mathbf{v}_1$

$\mathbf{v}_3 = \mathbf{v}_1 \times \mathbf{v}_2$